

# TWO POINT EIGENVALUE CORRELATION FOR A CLASS OF NON-SELFADJOINT OPERATORS UNDER RANDOM PERTURBATIONS

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**ABSTRACT.** We consider a non-selfadjoint  $h$ -differential model operator  $P_h$  in the semi-classical limit ( $h \rightarrow 0$ ) subject to random perturbations with a small coupling constant  $\delta$ . Assume that  $e^{-\frac{1}{C^h}} < \delta \ll h^\kappa$  for constants  $C, \kappa > 0$  suitably large. Let  $\Sigma$  be the closure of the range of the principal symbol.

We study the 2-point intensity measure of the random point process of eigenvalues of the randomly perturbed operator  $P_h^\delta$  and prove an  $h$ -asymptotic formula for the average 2-point density of eigenvalues. With this we show that two eigenvalues of  $P_h^\delta$  in the interior of  $\Sigma$  exhibit close range repulsion and long range decoupling.

**Résumé** Nous considérons un opérateur différentiel non-autoadjooint  $P_h$  dans la limite semiclassique ( $h \rightarrow 0$ ) soumis à de petites perturbations aléatoires. De plus, nous imposons que la constant de couplage  $\delta$  vérifie  $e^{-\frac{1}{C^h}} < \delta \ll h^\kappa$  pour certaines constantes  $C, \kappa > 0$  choisies assez grandes. Soit  $\Sigma$  l'adhérence de l'image du symbole principal de  $P_h$ .

Dans cet article, nous donnons une formule  $h$ -asymptotique pour la 2-points densité des valeurs propres en étudiant la mesure de comptage aléatoire des valeurs propres à l'intérieur de  $\Sigma$ . En étudiant cette densité, nous prouvons que deux valeurs propres sont répulsives à distance courte et indépendantes à long distance.

## 1. INTRODUCTION

It is well known that the norm of the resolvent of non-normal operators can be very large even far away from the spectrum. Consequently, the spectrum of such operators can be highly unstable even under tiny perturbations, cf [5, 7, 6, 17, 27]. A way to quantify this zone of spectral instability is given by the  $\varepsilon$ -pseudospectrum. Following the work of L.N. Trefethen and M. Embree [9], the  $\varepsilon$ -pseudospectrum of a closed linear operator  $A$  on a Banach space  $X$  is defined by

$$\sigma_\varepsilon(A) := \left\{ z \in \mathbb{C} \setminus \sigma(A); \|(z - A)^{-1}\| > \frac{1}{\varepsilon} \right\} \cup \sigma(A),$$

where  $\sigma(A)$  denotes the spectrum of  $A$ . Equivalently,

$$\sigma_\varepsilon(A) = \bigcup_{\substack{B \in \mathcal{B}(X) \\ \|B\| < \varepsilon}} \sigma(A + B). \quad (1.1)$$

In view of (1.1) it is natural to study the spectrum of such operators under small random perturbations. One line of recent interest has focused on the case of elliptic (pseudo-)differential operators subject to small random perturbations:

A series of papers by W. Bordeaux-Montrieux, M. Hager and J. Sjöstrand [11, 2, 10, 3, 13, 23, 24, 22] established a probabilistic Weyl law in the interior of the pseudospectrum for a large class of elliptic (pseudo-)differential operators subject to small random perturbations in the semiclassical or high energy limit. Furthermore, a similar result has been obtained by T. Christiansen and M. Zworski for certain randomly perturbed Toeplitz operators in [4].

In [28], we considered a class of elliptic semiclassical differential operators introduced by M. Hager [11] and obtained a precise  $h$ -asymptotic description of the average density of eigenvalues in the entire pseudospectrum by studying the first moment of linear statistics of the random point process of eigenvalues. In particular, we showed that there is an accumulation of eigenvalues in a small neighbourhood of the boundary of the pseudospectrum, leading to a break down of the Weyl law.

However, there have not yet been any results concerning the statistical correlation between the eigenvalues. The purpose of this paper is, therefore, to study the 2-point eigenvalue correlation in the case of Hager's model operator (cf. [11]):

**Hager's model operator.** Let  $0 < h \ll 1$ , we consider on  $S^1 = \mathbb{R}/2\pi\mathbb{Z}$  the semiclassical operator  $P_h : L^2(S^1) \rightarrow L^2(S^1)$  given by

$$P_h := hD_x + g(x), \quad D_x := \frac{1}{i} \frac{d}{dx}, \quad g \in C^\infty(S^1; \mathbb{C}) \quad (1.2)$$

where we assume that  $g \in C^\infty(S^1; \mathbb{C})$  is such that  $\text{Im } g$  has exactly two critical points and they are non-degenerate, one minimum and one maximum, say at  $a, b \in S^1$ , with  $\text{Im } g(a) < \text{Im } g(b)$ .

We denote the semiclassical principal symbol of  $P_h$  by

$$p(x, \xi) = \xi + g(x), \quad (x, \xi) \in T^*S^1. \quad (1.3)$$

The Poisson bracket of  $p$  and  $\bar{p}$  is given by

$$\{p, \bar{p}\} = p'_\xi \cdot \bar{p}'_x - p'_x \cdot \bar{p}'_\xi.$$

The spectrum of  $P_h$  is discrete with simple eigenvalues, given by

$$\sigma(P_h) = \{z \in \mathbb{C} : z = \langle g \rangle + kh, \ k \in \mathbb{Z}\}, \quad \langle g \rangle := (2\pi)^{-1} \int_0^{2\pi} g(y) dy. \quad (1.4)$$

**Zone of spectral instability.** For semiclassical pseudo-differential operators there are various ways to quantify the zone of spectral instability: Following [7, 29, 8], we define for  $p$  as in (1.3)

$$\Sigma := \overline{p(T^*S^1)} \subset \mathbb{C}.$$

In the case of (1.2) and (1.3)  $p(T^*S^1)$  is already closed due to the ellipticity of  $P_h$ . Next, for  $z \in \mathring{\Sigma}$ , consider the equation  $z = p(x, \xi)$ . It has precisely two solutions  $\rho_\pm := (x_\pm, \xi_\pm)$  where  $x_\pm$  are given by

$$\text{Im } g(x_\pm) = \text{Im } z, \text{ with } \pm \text{Im } g'(x_\pm) < 0 \quad (1.5)$$

and  $\xi_\pm = \text{Re } z - \text{Re } g(x_\pm)$ . Since we have that  $\{\text{Re } p, \text{Im } p\}(\rho_+(z)) < 0$  for all  $z \in \Omega \Subset \mathring{\Sigma}$  it follows from the work of N. Dencker, J. Sjöstrand and M. Zworski [8] that we can construct  $h^\infty$ -quasimodes  $u \in L^2(S^1)$  of  $P_h$  with semiclassical wave front set  $\text{WF}_h(u) = \{\rho_+(z)\}$  (in the case of (1.2) we can even construct exponentially accurate quasimodes even though

it is not analytic cf [11, 28]). We recall that for  $v = v(h)$ ,  $\|v\|_{L^2(S^1)} = \mathcal{O}(h^{-N})$ , for some fixed  $N$ , the semiclassical wave front set of  $v$  is defined by

$$\text{WF}_h(v) := \mathbb{C} \left\{ (x, \xi) \in T^*S^1 : \exists a \in \mathcal{S}(T^*S^1), a(x, \xi) = 1, \|a^w v\|_{L^2(S^1)} = \mathcal{O}(h^\infty) \right\}$$

where  $a^w$  denotes the Weyl quantization of  $a$ .

Alternatively, it has been shown in [11, 23, 28] that for all  $\Omega \Subset \mathring{\Sigma}$  and all  $z \in \Omega$

$$\|(P_h - z)^{-1}\| \geq C_1 e^{\frac{1}{C_2 h}},$$

with  $C_1, C_2 > 0$  constants that only depend on  $\Omega$ . This implies that such an  $\Omega$  is inside the  $e^{-1/C_2 h}$ -pseudospectrum of  $P_h$ .

Next, by the natural projection  $\Pi : \mathbb{R} \rightarrow S^1 = \mathbb{R}/2\pi\mathbb{Z}$  and a slight abuse of notation we identify the points  $x_\pm, a, b \in S^1$  with points  $x_\pm, a, b \in \mathbb{R}$  such that  $x_- - 2\pi < x_+ < x_-$  and  $b - 2\pi < a < b$ . Furthermore, we will identify  $S^1$  with the interval  $[b - 2\pi, b[$ .

**Adding a random perturbation.** We are interested in the following random perturbation of  $P_h$ :

$$P_h^\delta := P_h + \delta Q_\omega, \quad 0 \leq \delta \ll 1, \quad (1.6)$$

where  $Q_\omega$  is an integral operator  $L^2(S^1) \rightarrow L^2(S^1)$  of the form

$$Q_\omega u(x) := \sum_{|j|, |k| \leq \lfloor \frac{C_1}{h} \rfloor} \alpha_{j,k}(u|e^k) e^j(x). \quad (1.7)$$

Here,  $\lfloor x \rfloor := \max\{n \in \mathbb{N} : x \geq n\}$  for  $x \in \mathbb{R}$ ,  $C_1 > 0$  is large enough,  $e^k(x) := (2\pi)^{-1/2} e^{ikx}$ ,  $k \in \mathbb{Z}$ , and  $\alpha_{j,k}$  are complex valued independent and identically distributed random variables with complex Gaussian distribution law  $\mathcal{N}_{\mathbb{C}}(0, 1)$ . Since,  $Q_\omega$  is a compact operator, the spectrum of  $P_h^\delta$  is discrete.

We recall that a random variable  $\alpha$  has complex Gaussian distribution law  $\mathcal{N}_{\mathbb{C}}(0, 1)$  if

$$\alpha_*(P(d\omega)) = \frac{1}{\pi} e^{-\alpha \bar{\alpha}} L(d\alpha)$$

where  $L(d\alpha)$  denotes the Lebesgue measure on  $\mathbb{C}$  and  $\omega$  is the random parameter living in the sample space  $\mathcal{M}$  of a probability space  $(\mathcal{M}, \mathcal{A}, P)$  with  $\sigma$ -algebra  $\mathcal{A}$  and probability measure  $P$ .  $\alpha \sim \mathcal{N}_{\mathbb{C}}(0, 1)$  implies that  $\alpha$  has expectation 0 and variance 1, since

$$\mathbb{E}[\alpha] = 0, \quad \text{and} \quad \mathbb{E}[|\alpha|^2] = 1.$$

Here,  $\mathbb{E}[\cdot]$  denotes the expectation. The Markov inequality yields that for  $C > 0$  large enough

$$\|Q_\omega\|_{\text{HS}} \leq \frac{C}{h}, \quad \text{with probability} \leq 1 - e^{-\frac{1}{h^2}}. \quad (1.8)$$

This, has been obtained as well by W. Bordeaux-Montrieux in [2]. Hence, we restrict our probability space to a open ball  $B(0, R) \Subset \mathbb{C}^N$ , with  $N := (2 \lfloor \frac{C_1}{h} \rfloor + 1)^2$ , of radius  $R = C/h$  and centered at 0, to obtain a uniform (in the random variables) bound on  $Q_\omega$ .

In this paper we are interested in the eigenvalues of  $P_h^\delta$  in the interior of the pseudospectrum. Therefore, we make the following assumptions on  $\Omega \Subset \Sigma$ :

**Hypothesis 1.** *We assume that there exists a  $C > 1$  such that*

$$\Omega \Subset \overset{\circ}{\Sigma} \text{ is open, convex, relatively compact and simply connected with } \text{dist}(\Omega, \partial\Sigma) > \frac{1}{C}. \quad (1.9)$$

It will be very useful to give bounds on the coupling constant  $\delta$  in terms of the imaginary part of the action between  $\rho_+(z)$  and  $\rho_-(z)$ ,  $z \in \Omega$  as in (1.9) (cf (1.5)), defined by:

$$S := \min \left( \text{Im} \int_{x_+}^{x_-} (z - g(y)) dy, \text{Im} \int_{x_+}^{x_- - 2\pi} (z - g(y)) dy \right). \quad (1.10)$$

**Hypothesis 2.** *The coupling constant  $\delta > 0$  in (1.6) satisfies*

$$\delta := \delta(h) := \sqrt{h} e^{-\frac{\epsilon_0(h)}{h}} \quad (1.11)$$

with  $(\kappa - \frac{1}{2}) h \ln(h^{-1}) + Ch \leq \epsilon_0(h) < \min_{z \in \overline{\Omega}} S(z)/C$  for some  $\kappa > 52/10$  and  $C > 0$  large and where the last inequality is uniform in  $h > 0$ . Equivalently,  $\delta$  satisfies the inequality

$$\sqrt{h} \exp \left\{ -\frac{\min_{z \in \overline{\Omega}} S(z)}{Ch} \right\} < \delta \ll h^\kappa.$$

**Remark 3.** *We chose these hypotheses because the aim of this paper is to treat the two-point eigenvalue density and correlation in the interior of the pseudospectrum. Hypotheses 1 and 2 prevent us from reaching the pseudospectral boundary since either we need to allow for sufficiently small coupling constants which would bring the boundary of the pseudospectrum in the interior of  $\Omega$  (with  $\text{dist}(\Omega, \partial\Sigma) > 1/C$  for some  $C > 0$ ), or we need to allow sets  $\Omega \Subset \Sigma$  with  $\text{dist}(\Omega, \partial\Sigma) \geq Ch^{2/3}$ . The two-point interaction close to the pseudospectral boundary remains an interesting open problem.*

## 2. MAIN RESULTS

We are interested in the 2-point correlation of eigenvalues of the perturbed operator  $P_h^\delta$ . Therefore, we study the 2-point intensity measure  $\nu$ , given by

$$\mathbb{E} \left[ \sum_{\substack{z, w \in \sigma(P_h^\delta) \\ z \neq w}} \varphi(z, w) \mathbb{1}_{B(0, R)} \right] = \int_{\mathbb{C}^2} \varphi(z, w) d\nu(z, w), \quad \varphi \in \mathcal{C}_0(\Omega^2). \quad (2.1)$$

**Remark 4.** *The above approach is more classical in the study of zeros of random polynomials and Gaussian analytic functions; we refer the reader to the works of B. Shiffman and S. Zelditch [20, 21, 19, 18], M. Sodin [26] and the book [14] by J. Hough, M. Krishnapur, Y. Peres and B. Virág.*

We begin by giving an  $h$ -asymptotic formula for its Lebesgue density valid at a distance  $\gg h^{3/5}$  from the diagonal. For  $\Omega$  as in (1.9) and  $C_2 > 0$ , we define the set

$$D_h(\Omega, C_2) := \{(z, w) \in \Omega^2; |z - w| \leq C_2 h^{3/5}\}. \quad (2.2)$$

Before, we state the main result, let us recall that has been shown in [11, 28] that the direct image  $p_*(d\xi \wedge dx)$  of the symplectic volume form  $d\xi \wedge dx$  on  $T^*S^1$  is absolutely

continuous with respect to the Lebesgue measure on  $\mathbb{C}$  and its Radon-Nikodym derivative is

$$\sigma(z) := \frac{p_*(d\xi \wedge dx)}{L(dz)} = \left( \frac{2i}{\{p, \bar{p}\}(\rho_+(z))} + \frac{2i}{\{\bar{p}, p\}(\rho_-(z))} \right). \quad (2.3)$$

**Theorem 5.** *Let  $\Omega \Subset \Sigma$  be as in (1.9). Let  $\delta > 0$  be as in Hypothesis 2. Let  $\nu$  be the measure defined in (2.1) and let  $\sigma(z)$  be as in (2.3). Then, for  $|z - w| \leq 1/C$  with  $C > 1$  large enough, there exist smooth functions*

- $\sigma_h(z, w) = \sigma\left(\frac{z+w}{2}\right) + \mathcal{O}(h)$ ,
- $K(z, w; h) = \sigma_h(z, w) \frac{|z-w|^2}{4h} (1 + \mathcal{O}(|z-w| + h^\infty))$ ,
- $D^\delta(z, w; h) = \frac{\Lambda(z, w)}{(2\pi h)^2(1-e^{-2K})} \left(1 + \mathcal{O}\left(\delta h^{-\frac{8}{5}}\right)\right) + \mathcal{O}\left(e^{-\frac{D}{h^2}}\right)$ , with

$$\begin{aligned} \Lambda(z, w; h) = & \sigma_h(z, z)\sigma_h(w, w) + \sigma_h(z, w)^2(1 + \mathcal{O}(|z-w|))e^{-2K} \\ & + \frac{\sigma_h(z, w)^2(1 + \mathcal{O}(|z-w|))}{e^K \sinh(K)} (2K^2 \coth(K) - 4K) + \mathcal{O}\left(h^\infty + \delta h^{-\frac{32}{10}}\right) \end{aligned}$$

and there exists a constant  $c > 0$  such that for all  $\varphi \in \mathcal{C}_0^\infty(\Omega^2 \setminus D_h(\Omega, c))$

$$\int_{\mathbb{C}^2} \varphi(z, w) d\nu(z, w) = \int_{\mathbb{C}^2} \varphi(z, w) D^\delta(z, w; h) L(d(z, w)).$$

By this result we see that in the interior of the pseudospectrum the leading terms of the 2-point density of eigenvalues depends only on the symplectic volume form in phase space. This agrees very well with previous results of Hager, Sjöstrand and Vogel [11, 23, 28] saying that in the interior of the pseudospectrum the probabilistic and average density of eigenvalues depends only on the symplectic volume form.

Let us stress once more that due to the assumptions on  $\Omega$  and  $\delta$ , the formula for the 2-point density presented in Theorem 5 is not valid close to the pseudospectral boundary. However, in view of the results presented in [28], we would expect the 2-point density to change drastically close to the pseudospectral boundary, but for now this remains an open problem.

**Remark 6.** *Avoiding  $D_h(\Omega, c)$ , cf. (2.2), with the support of the test functions  $\varphi$  in Theorem 5 is due to a technical difficulty in the proof, since there is some degeneracy due to the error terms when  $|z - w|$  is too small, see Proposition 35 below.*

*However, having a formula for the 2-point density of eigenvalues outside  $D_h(\Omega, c)$  is sufficient to include the study of the close range correlation between two eigenvalues up to a certain distance, cf. Theorem 7 and 8.*

**2.1. Asymptotic regimes of the density.** Using the formula obtained in Theorem 5, we will prove that two eigenvalues of  $P_h^\delta$  exhibit the following interaction:

**Theorem 7.** *Under the hypothesis of Theorem 5, we have that*

- for  $h^{\frac{4}{7}} \ll |z - w| \ll h^{\frac{1}{2}}$

$$D^\delta(z, w; h) = \frac{\sigma_h^3(z, w)|z - w|^2}{(4\pi)^2 h^3} \left(1 + \mathcal{O}\left(\frac{|z - w|^2}{h} + \delta h^{-\frac{8}{5}}\right)\right);$$

- for  $|z - w| \gg (h \ln h^{-1})^{\frac{1}{2}}$

$$D^\delta(z, w; h) = \frac{\sigma(z)\sigma(w) + \mathcal{O}(h)}{(2h\pi)^2} \left(1 + \mathcal{O}\left(\delta h^{-\frac{8}{5}}\right)\right).$$

Let us give some comments on this result: The fact that we cannot analyze the eigenvalue interaction completely up to the diagonal is due to some technical difficulties. In the above theorem, two eigenvalues of the perturbed operator  $P_h^\delta$  show the following types of interaction:

**Short range repulsion:** The two-point density decays quadratically in  $|z - w|$  if two eigenvalues are too close, and in view of the numerical simulations presented in Section 2.4 we conjecture that this is the case for all  $z, w$  as above satisfying  $0 < |z - w| \ll h^{\frac{1}{2}}$ .

**Long range decoupling:** If the distance between two eigenvalues is  $\gg (h \ln h^{-1})^{\frac{1}{2}}$  the two-point density is given by the product of two one-point densities (cf. (2.6)). This means that at this distance two eigenvalues are placed in average in an uncorrelated way.

**2.2. 2-point correlation function.** M. Hager [11] showed, using subharmonic estimates, that, with probability close to 1, the eigenvalues of the perturbed operator  $P_h^\delta$  contained in  $\Omega$  (as in Hypothesis 1) follow a Weyl law, i.e.

$$\#(\sigma(P_h^\delta) \cap \Omega) \sim \frac{1}{2\pi h} \text{vol}(\{\rho \in T^*S^1; p(\rho) \in \Omega\}).$$

In [28], we considered the random point process given by eigenvalues of  $P_h^\delta$ :

$$\Xi := \sum_{z \in \sigma(P_h^\delta)} \delta_z. \quad (2.4)$$

where the eigenvalues are counted according to their multiplicities and  $\delta_z$  denotes the Dirac-measure at  $z$ .

We studied in [28] the first moment of linear statistics of  $\Xi$  with the random variables  $\alpha$  restricted to a ball  $B(0, R) \subset \mathbb{C}^N$  with  $R = C/h$ , i.e. the measure  $\mu_1$  defined by

$$\mathbb{E}[\Xi(\varphi) \mathbf{1}_{B(0, R)}] = \int_{\mathbb{C}} \varphi(z) d\mu_1(z)$$

for all  $\varphi \in \mathcal{C}_0(\Omega)$  with  $\Omega \Subset \Sigma$  such that  $\text{dist}(\Omega, \partial\Sigma) \gg h^{2/3}$ . For  $\Omega$  as in Hypothesis 1, Theorem 2.11 in [28] implies that

$$\mathbb{E}[\Xi(\varphi) \mathbf{1}_{B(0, R)}] = \int \varphi(z) d(z; h) L(dz), \quad \forall \varphi \in \mathcal{C}_0(\Omega), \quad (2.5)$$

where

$$d(z; h) = \frac{1}{2\pi h} \sigma(z) + \mathcal{O}(1), \quad \sigma(z) \text{ is as in (2.3)}. \quad (2.6)$$

In other words, the average density of eigenvalues in  $\Omega$  is up to first order determined by symplectic volume form in phase space.

It follows from (2.6), (2.3) that for  $h > 0$  small enough  $d(z; h) > 0$  for all  $z \in \Omega$  as in (1.9). Hence, under the assumptions of Theorem 5, the 2-point correlation function of the eigenvalues of  $P_h^\delta$ , is well defined and given by

$$\kappa^\delta(z, w; h) := \frac{D^\delta(z, w; h)}{d(w; h)d(z; h)}.$$

**Theorem 8.** *Under the hypothesis of Theorem 5, we have that for  $(z, w) \in \Omega^2 \setminus D_h(\Omega, c)$  as in Theorem 5 that*

$$\begin{aligned} \kappa^\delta(z, w; h) = & \frac{1 + \mathcal{O}(h)}{(1 - e^{-2K})} \left( 1 + (1 + \mathcal{O}(|z - w|))e^{-2K} + \frac{(1 + \mathcal{O}(|z - w|))}{e^K \sinh(K)} (2K^2 \coth(K) - 4K) \right. \\ & \left. + \mathcal{O}\left(h^\infty + \delta h^{-\frac{32}{10}}\right) \right) + \mathcal{O}\left(e^{-\frac{D}{h^2}}\right). \end{aligned}$$

Moreover, we have the following asymptotic behaviour of the 2-point correlation function  $\kappa^\delta(z, w; h)$ :

- for  $h^{\frac{4}{7}} \ll |z - w| \ll h^{\frac{1}{2}}$

$$\kappa^\delta(z, w; h) = \frac{\sigma_h(z, w)|z - w|^2}{4h} \left( 1 + \mathcal{O}\left(\frac{|z - w|^2}{h} + \delta h^{-\frac{8}{5}}\right) \right) \ll 1;$$

- for  $|z - w| \gg (h \ln h^{-1})^{\frac{1}{2}}$

$$\kappa^\delta(z, w; h) = 1 + \mathcal{O}(h).$$

In the above Theorem we see that two eigenvalues of  $P_h^\delta$  shows the following behaviour:

**Short range repulsion:** The 2-point correlation function  $\kappa^\delta(z, w; h)$  decays quadratically in  $|z - w|$  if the distance between  $z$  and  $w$  is smaller than a term of order  $h^{\frac{1}{2}}$ . It is thus less likely to find two eigenvalues close together. Furthermore, we see by (2.3) that  $\sigma(z)$  grows towards the boundary of  $\Sigma$ , hence the short range repulsion is weaker for  $\Omega$  closer to the boundary of  $\Sigma$ , as we expected from the numerical simulations presented in [28], see Figure 4 therein.

The fact that we cannot analyze close range correlation up to the diagonal is due to a degeneracy resulting from error terms, cf. Remark 6 and the proofs of Theorem 8 and 7. However, the conclusions of Theorem 8 allow for the study of the scaling limit of the 2-point correlation function, which yields the limiting local 2-point statistics of eigenvalues of  $P_h^\delta$ , after re-scaling distances between eigenvalues to be independent of  $h$ , cf. Section 2.3 and Corollary 10.

**Long range decoupling:** If the distance between  $z$  and  $w$  is larger than a term of order  $(h \ln h^{-1})^{\frac{1}{2}}$ , the 2-point correlation function  $\kappa^\delta(z, w; h)$  is given up to a small error by 1. Hence, we see that at these distances two eigenvalues of  $P_h^\delta$  are up to a small error uncorrelated.

**Remark 9.** Recall from the discussion after (1.6) that in this paper we focus on the case where the random perturbation is given by a random matrix whose entries are independent and identically distributed complex Gaussian random variables. As supported by numerical experiments (cf. Section 2.4) we expect Theorem 8 to hold for a much more general class of random variables as long as the perturbation is of the form (1.7).

Questions concerning the universality of the result of Theorem 8 in the case of small random perturbations of a more general class of (pseudo-)differential operators are currently under investigation by the author. We expect the type of perturbation (by random matrix or by random potential) rather than its probability distribution to be decisive, since although in both cases we can obtain a probabilistic Weyl law for the eigenvalues, see [10, 13, 24, 22], numerical experiments suggest that the 2-point correlation functions in both cases differ.

**2.3. Scaling limit of the 2-point correlation function.** We can use Theorem 8 to study the limiting local 2-point correlation function in the interior of the pseudospectrum. Therefore, let  $\Omega$  be as in (1.9), and fix a  $z_0 \in \Omega$ . Let  $d(z; h)$  be as in (2.6) and set  $d_0 := d(z_0; h) \asymp h^{-1}$ . Let  $\kappa^\delta(z, w; h)$  be as in Theorem 8, let  $W$  be a compact subset of  $\{(z, w) \in \mathbb{C}^2; z \neq w\}$  and consider, for  $h > 0$  small enough,

$$\tilde{\kappa}_h(z, w) := \kappa^\delta(z_0 + d_0^{-1/2}z, z_0 + d_0^{-1/2}w; h), \quad (z, w) \in W. \quad (2.7)$$

This is well defined since for  $h > 0$  small enough  $(z_0 + d_0^{-1/2}z, z_0 + d_0^{-1/2}w) \in \Omega^2 \setminus D_h(\Omega, c)$  (see Theorem 8) for all  $(z, w) \in W$ . Similarly to the discussion before Theorem 8, we notice that we can view  $\tilde{\kappa}_h(z, w)$  as the 2-point correlation function of the random point process of the re-scaled eigenvalues of  $P_h^\delta$ :

$$\tilde{\Xi} := \sum_{z \in \sigma(P_h^\delta)} \delta_{(z - z_0)d_0^{1/2}}. \quad (2.8)$$

When considering the first moment of linear statistics of  $\tilde{\Xi}$  (cf. (2.5), (2.6)) we see that we have re-scaled distances in such a way that the leading order of the average density of eigenvalues (after re-scaling) is independent of  $h$ .

From Theorem 8 we obtain the following result.

**Corollary 10.** *For any compact  $W \subseteq \{(z, w) \in \mathbb{C}^2; z \neq w\}$  we have that*

$$\lim_{h \rightarrow 0^+} \tilde{\kappa}_h(z, w) = \kappa\left(\frac{\pi}{2}|z - w|^2\right), \quad (z, w) \in W,$$

*uniformly on  $W$ , where*

$$\kappa(t) = \frac{(\sinh^2 t + t^2) \cosh t - 2t \sinh t}{\sinh^3 t}, \quad t = \frac{\pi}{2}|z - w|^2. \quad (2.9)$$

Let us remark that the scaling limit 2-point correlation function is independent of  $z_0$  and depends only on the distance between points. Similar to the asymptotic regimes presented in Theorem 8, we obtain short range repulsion between two re-scaled eigenvalues of  $P_h^\delta$  since, by Taylor expansion,  $\kappa(t) = t(1 + \mathcal{O}(t^2))$ , as  $t \rightarrow 0^+$ , which shows that the scaling limit 2-point correlation function decays quadratically for small distances between 2 points (see Figure 1).

Similarly, we have long range decorrelation between two re-scaled eigenvalues of  $P_h^\delta$  since, by Taylor expansion,  $\kappa(t) = 1 + \mathcal{O}(t^2 e^{-2t})$ , as  $t \rightarrow +\infty$ .

The same scaling limit 2-point correlation function  $\kappa$  has been found as well by J.H. Hannay [12] in the case of zeros of certain random polynomials and by P. Bleher, B. Shiffman and S. Zelditch [1] in the case of random holomorphic sections of the  $N$ th power of a positive Hermitian line bundle over a compact complex manifold.



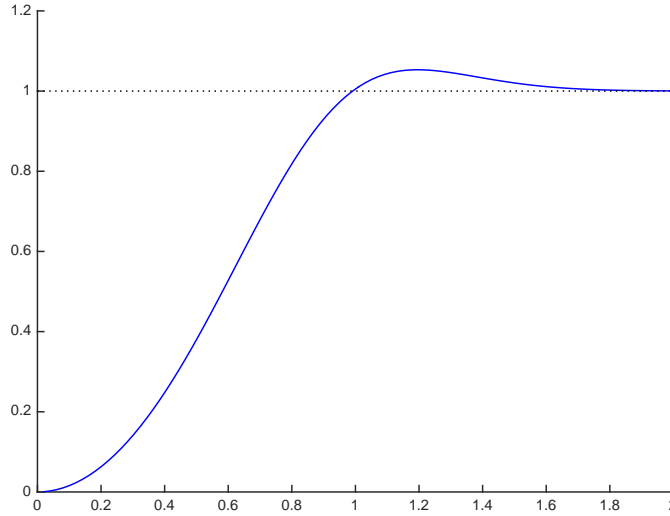


FIGURE 1. The pair correlation function  $\kappa(\frac{\pi}{2}|z - w|^2)$ , as a function of the distance, as given in Corollary 10. A similar figure can be found in [12, 1]

**2.4. Numerical simulation.** To illustrate Theorem 8 and Corollary 10, we have numerically determined the 2-point correlation function of the eigenvalues of a discretisation of the operator  $hD_x + e^{-ix}$ , with  $h = 2 \cdot 10^{-3}$ , perturbed with a random complex Gaussian matrix with coupling constant  $\delta = 2 \cdot 10^{-12}$ . The left hand side of Figure 2 shows one realisation of these eigenvalues and the region, where we determine the 2-point correlation function, staying inside of the pseudospectrum and away from the effects caused by the finite dimensional approximation of the operator. The right hand side shows the eigenvalues in the region of interest after re-scaling by  $z \mapsto d(0; h)^{1/2}z$ , as in (2.8).

Figure 3 compares the scaling limit pair correlation function  $\kappa(\frac{\pi}{2}|z - w|^2)$  (as a function of the distance) to the histogram data of the numerically obtained re-scaled 2-point correlation function, which corresponds to  $\tilde{\kappa}_h(z, w)$  as in (2.8), obtained from the numerically simulated re-scaled eigenvalues depicted on the right hand side of Figure 2 and averaged over 200 realisations of Gaussian random matrices.

We see that up to a small error the numerically determined re-scaled 2-point correlation function is given by its scaling limit, showing decorrelation for large distances and quadratic decay, as the distance between two points goes to zero, confirming the conclusions of Theorem 8 and Corollary 10.

Finally, let us remark, that when running numerical experiments with a perturbation given by a complex random matrix whose entries follow a uniform or a Poisson distribution instead of a complex Gaussian one, we are able to produce the same results as presented in Figure 3, suggesting that the results of Theorem 8 and Corollary 10 are valid for random perturbations of the form 1.7 given by a more general class of random variables.

**Organisation of this paper.** In Section 3 we recall some results from [28] needed for this paper and we provide a formula (cf. Proposition 17) representing the two-point density

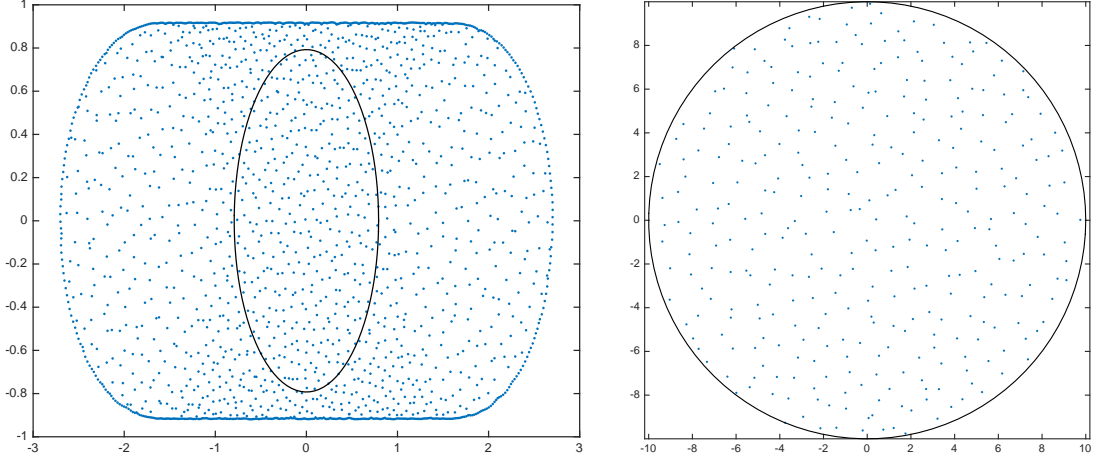


FIGURE 2. On the left hand side we present the spectrum of the discretisation of  $hD + \exp(-ix)$ ,  $h = 2 \cdot 10^{-3}$ , (approximated by a  $2001 \times 2001$ -matrix) perturbed with a random complex Gaussian matrix with coupling constant  $\delta = 2 \cdot 10^{-12}$ . The black disc indicates the region where we determine the 2-point correlation function presented on the right hand side of Figure 3. The right hand side shows the same disc after re-scaling by  $d(0; h) \approx (\pi h)^{-1}$ , the average density of eigenvalues at 0, cf. (2.6).

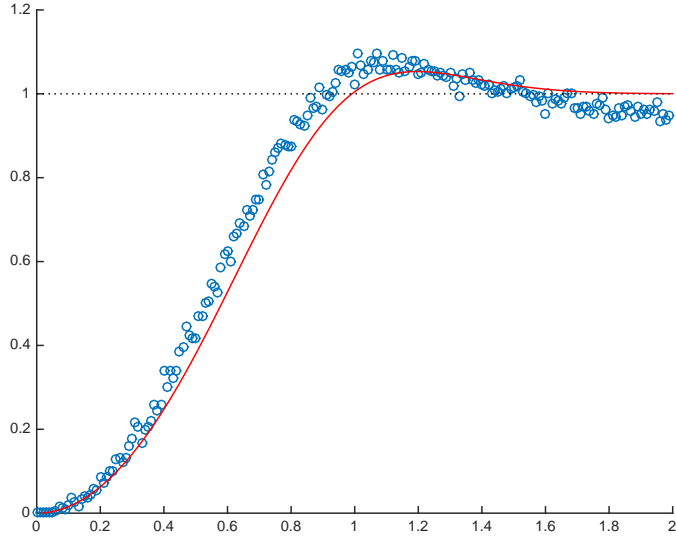


FIGURE 3. The red line shows the scaling limit pair correlation function  $\kappa(\frac{\pi}{2}|z - w|^2)$ , as a function of the distance, and the blue circles show the histogram data corresponding to the numerically determined 2-point correlation function given by the re-scaled eigenvalues of the random matrix presented in Figure 2.

of eigenvalues in terms of the permanent and determinant of certain correlation matrices. This formula will be proved in Section 6. Section 4 provides a detailed description of the elements of these matrices using the method of stationary phase. Section 5 then exploits the main result of Section 4 to obtain precise formulas and estimates for the permanent and determinant of the matrices appearing in Proposition 17. Section 7 states the proofs of the main results of this paper.

**Notation.** We will use the standard scalar products on  $L^2(S^1)$  and  $\mathbb{C}^N$  defined by

$$(f|g) := \int_{S^1} f(x)\overline{g(x)}dx, \quad f, g \in L^2(S^1),$$

and

$$(X|Y) := \sum_{i=1}^N X_i \overline{Y_i}, \quad X, Y \in \mathbb{C}^N.$$

Throughout this work we shall denote the Lebesgue measure on  $\mathbb{C}$  by  $L(dz)$ ; denote  $d(z) := \text{dist}(z, \partial\Sigma)$ ; work with the convention that when we write  $\mathcal{O}(1)^{-1}$  then we mean implicitly an arbitrarily small positive constant; denote by  $f(x) \asymp g(x)$  that there exists a constant  $C > 0$  such that  $C^{-1}g(x) \leq f(x) \leq Cg(x)$ .

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### 3. A FORMULA FOR THE TWO-POINT INTENSITY MEASURE

In this section we will give a short review of a well-posed Grushin problem for the perturbed operator  $P_h^\delta$  which has already been used in [28, 23]. We will then employ the resulting effective Hamiltonians to derive a formula for the two-point intensity measure defined in (2.1).

We recall that we always suppose that  $\Omega \Subset \overset{\circ}{\Sigma}$  is such that Hypothesis 1 is satisfied, if nothing else is specified.

**3.1. Grushin Problem.** We begin by giving a short refresher on Grushin problems. They have become an important tool in microlocal analysis and are employed with great success in a vast number of works. As reviewed in [25], the central idea is to set up an auxiliary problem of the form

$$\begin{pmatrix} P(z) & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_- \longrightarrow \mathcal{H}_2 \oplus \mathcal{H}_+,$$

where  $P(z)$  is the operator under investigation and  $R_\pm$  are suitably chosen. We say that the Grushin problem is well-posed if this matrix of operators is bijective. If  $\dim \mathcal{H}_- = \dim \mathcal{H}_+ < \infty$ , one typically writes

$$\begin{pmatrix} P(z) & R_- \\ R_+ & 0 \end{pmatrix}^{-1} = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix}.$$

The key observation goes back to the Shur complement formula or, equivalently, the Lyapunov-Schmidt bifurcation method, i.e. the operator  $P(z) : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  is invertible if and only if the finite dimensional matrix  $E_{-+}(z)$  is invertible and when  $E_{-+}(z)$  is invertible, we have

$$P^{-1}(z) = E(z) - E_+(z)E_{-+}^{-1}(z)E_-(z).$$

$E_{-+}(z)$  is sometimes called effective Hamiltonian.

Next, we give a short reminder of the Grushin Problem used to study  $P_h^\delta$ . First, we introduce the following auxiliary operators which have already been used by M. Hager J. Sjöstrand in [13].

**3.2. Two auxiliary operators.** For  $z \in \mathbb{C}$  we consider  $Q(z)$  and  $\tilde{Q}(z)$ , two  $z$ -dependent elliptic self-adjoint operators from  $L^2(S^1)$  to  $L^2(S^1)$ , defined by

$$Q(z) := (P_h - z)^*(P_h - z), \quad \tilde{Q}(z) := (P_h - z)(P_h - z)^* \quad (3.1)$$

with natural domains given by  $\mathcal{D}(Q(z)), \mathcal{D}(\tilde{Q}(z)) = H_{\text{sc}}^2(S^1)$ . Since  $S^1$  is compact and these are elliptic, non-negative, self-adjoint operators their spectra are discrete and contained in the interval  $[0, \infty[$ . Since

$$Q(z)u = 0 \Rightarrow (P_h - z)u = 0$$

it follows that  $\mathcal{N}(Q(z)) = \mathcal{N}(P_h - z)$  and  $\mathcal{N}(\tilde{Q}(z)) = \mathcal{N}((P_h - z)^*)$ . Furthermore, if  $\lambda \neq 0$  is an eigenvalue of  $Q(z)$  with corresponding eigenvector  $e_\lambda$  we see that  $f_\lambda := (P_h - z)e_\lambda$  is an eigenvector of  $\tilde{Q}(z)$  with the eigenvalue  $\lambda$ . Similarly, every non-vanishing eigenvalue of  $\tilde{Q}(z)$  is an eigenvalue of  $Q(z)$  and moreover, since  $P_h - z, (P_h - z)^*$  are Fredholm operators of index 0 we see that  $\dim \mathcal{N}(P_h - z) = \dim \mathcal{N}((P_h - z)^*)$ . Hence the spectra of  $Q(z)$  and  $\tilde{Q}(z)$  are equal

$$\sigma(Q(z)) = \sigma(\tilde{Q}(z)) = \{t_0^2, t_1^2, \dots\}, \quad 0 \leq t_j \nearrow \infty. \quad (3.2)$$

Now consider the orthonormal basis of  $L^2(S^1)$

$$\{e_0, e_1, \dots\} \quad (3.3)$$

consisting of the eigenfunctions of  $Q(z)$ . By the previous observations we have

$$(P_h - z)(P_h - z)^*(P_h - z)e_j = t_j^2(P_h - z)e_j.$$

Thus defining  $f_0$  to be the normalized eigenvector of  $\tilde{Q}$  corresponding to the eigenvalue  $t_0^2$  and the vectors  $f_j \in L^2(S^1)$ , for  $j \in \mathbb{N}^*$ , as the normalization of  $(P_h - z)e_j$  such that

$$(P_h - z)e_j = \alpha_j f_j, \quad (P_h - z)^* f_j = \beta_j e_j \quad \text{with } \alpha_j \beta_j = t_j^2, \quad (3.4)$$

yields an orthonormal basis of  $L^2(S^1)$

$$\{f_0, f_1, \dots\} \quad (3.5)$$

consisting of the eigenfunctions of  $\tilde{Q}(z)$ . Since  $\alpha_j = ((P_h - z)e_j | f_j) = (e_j | (P_h - z)^* f_j) = \bar{\beta}_j$  we can conclude that  $\alpha_j \bar{\alpha}_j = t_j^2$ .

**3.3. A Grushin Problem for the perturbed operator  $P_h^\delta$ .** Following Sjöstrand in [23], we use the eigenfunctions of the operators  $Q$  and  $\tilde{Q}$  (cf (3.1)) to create a well-posed Grushin Problem. The sequel is taken from [28], but it originates partly in the works of Hager [11], Bordeaux-Montrieux [2] and Sjöstrand [23].

**Proposition 11.** *Let  $z \in \Omega \Subset \Sigma$  with  $\text{dist}(\Omega, \partial\Sigma) > 1/C$  and let  $\alpha_0, e_0$  and  $f_0$  be as in (3.4). Define*

$$\begin{aligned} R_+ &: H^1(S^1) \longrightarrow \mathbb{C} : u \longmapsto (u|e_0), \\ R_- &: \mathbb{C} \longrightarrow L^2(S^1) : u_- \longmapsto u_- f_0. \end{aligned}$$

Then

$$\mathcal{P}(z) := \begin{pmatrix} P_h - z & R_- \\ R_+ & 0 \end{pmatrix} : H^1(S^1) \times \mathbb{C} \longrightarrow L^2(S^1) \times \mathbb{C}$$

is bijective with the bounded inverse

$$\mathcal{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix}$$

where  $E_-(z)v = (v|f_0)$ ,  $E_+(z)v_+ = v_+ e_0$ ,  $E(z) = (P_h - z)^{-1}|_{(f_0)^\perp \rightarrow (e_0)^\perp}$  and  $E_{-+}(z)v_+ = -\alpha_0 v_+$ . Furthermore, we have the estimates for  $z \in \Omega$

$$\begin{aligned} \|E_-(z)\|_{L^2 \rightarrow \mathbb{C}}, \|E_+(z)\|_{\mathbb{C} \rightarrow H^1} &= \mathcal{O}(1), \\ \|E(z)\|_{L^2 \rightarrow H^1} &= \mathcal{O}(h^{-1/2}), \\ |E_{-+}(z)| &= \mathcal{O}\left(\sqrt{h}e^{-\frac{s}{h}}\right) = \mathcal{O}\left(e^{-\frac{1}{Ch}}\right); \end{aligned} \tag{3.6}$$

**Definition 12.** For  $x \in \mathbb{R}$  we denote the integer part of  $x$  by  $\lfloor x \rfloor$ . Let  $C_1 > 0$  be big enough as above and define  $N := (2\lfloor \frac{C_1}{h} \rfloor + 1)^2$ . Let  $e_0$  and  $f_0$  be as in (3.4), let  $z \in \Omega \Subset \Sigma$  and let  $\widehat{e}_0(z; \cdot)$  and  $\widehat{f}_0(z; \cdot)$  denote the Fourier coefficients of  $e_0$  and  $f_0$ . We define the vector  $X(z) = (X_{j,k}(z))_{|j|, |k| \leq \lfloor \frac{C_1}{h} \rfloor} \in \mathbb{C}^N$  to be given by

$$X_{j,k}(z) = \widehat{e}_0(z; k) \overline{\widehat{f}_0(z; j)}, \quad \text{for } |j|, |k| \leq \left\lfloor \frac{C_1}{h} \right\rfloor. \tag{3.7}$$

**Proposition 13.** Let  $z \in \Omega \Subset \Sigma$ . Let  $N$  be as in Definition 12 and let  $B(0, R) \subset \mathbb{C}^N$  be the ball of radius  $R := C/h$ ,  $C > 0$  large, centered at 0. Let  $P_h^\delta$  be as in (1.6), (1.2). Let  $R_-, R_+$  be as in Proposition 11. Then

$$\mathcal{P}_\delta(z) := \begin{pmatrix} P_h^\delta - z & R_- \\ R_+ & 0 \end{pmatrix} : H^1(S^1) \times \mathbb{C} \longrightarrow L^2(S^1) \times \mathbb{C}$$

is bijective with the bounded inverse

$$\mathcal{E}_\delta(z) = \begin{pmatrix} E_\delta^+(z) & E_\delta^-(z) \\ E_\delta^-(z) & E_\delta^+(z) \end{pmatrix}$$

where

$$\begin{aligned} E^\delta(z) &= E(z) + \mathcal{O}(\delta h^{-2}) = \mathcal{O}(h^{-1/2}) \\ E_-^\delta(z) &= E_-(z) + \mathcal{O}(\delta h^{-3/2}) = \mathcal{O}(1) \\ E_+^\delta(z) &= E_+(z) + \mathcal{O}(\delta h^{-3/2}) = \mathcal{O}(1) \end{aligned}$$

and

$$E_{-+}^\delta(z) = E_{-+}(z) - \delta X(z) \cdot \alpha + T(z; \alpha), \quad (3.8)$$

with  $X(z) \cdot \alpha = E_- Q_\omega E_+$ ,  $\alpha \in B(0, R)$ , and

$$T(z, \alpha) := \sum_{n=1}^{\infty} (-\delta)^{n+1} E_- Q_\omega (E Q_\omega)^n E_+ = \mathcal{O}(\delta^2 h^{-5/2}). \quad (3.9)$$

Here, the dot-product  $X(z) \cdot \alpha$  is the natural bilinear one.

**Remark 14.** The effective Hamiltonian  $E_{-+}^\delta(z)$  depends smoothly on  $z \in \Omega$  and holomorphically on  $\alpha \in B(0, R) \subset \mathbb{C}^N$ . As in [28, (8.6) and Proposition 4.6] we have the following estimates: for all  $z \in \Omega$ , all  $\alpha \in B(0, R)$  and all  $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$

$$\begin{aligned} \partial_z^{\beta_1} \partial_{\bar{z}}^{\beta_2} E_{-+}(z) &= \mathcal{O}\left(h^{-|\beta|+1/2} e^{-\frac{S}{h}}\right), \text{ and} \\ \partial_z^{\beta_1} \partial_{\bar{z}}^{\beta_2} T(z, \alpha) &= \mathcal{O}\left(\delta^2 h^{-(|\beta|+\frac{5}{2})}\right) \end{aligned}$$

where  $S$  is as in (1.10).

Moreover, as remarked in [23] the effective Hamiltonian  $E_{-+}^\delta(z)$  satisfies a  $\bar{\partial}$ -equation, i.e. there exists a smooth function  $f^\delta : \Omega \rightarrow \mathbb{C}$  such that

$$\partial_{\bar{z}} E_{-+}^\delta(z) + f^\delta(z) E_{-+}^\delta(z) = 0.$$

This implies that the zeros of  $E_{-+}^\delta(z)$  are isolated and countable and we may use the same notion of multiplicity as for holomorphic functions.

**3.4. Counting zeros.** By the above well-posed Grushin Problem for the perturbed operator  $P_h^\delta$  we have that  $\sigma(P_h^\delta) = (E_{-+}^\delta)^{-1}(0)$ . Hence, to study the two-point intensity measure  $\nu$  defined in (2.1), we investigate the integral

$$\pi^{-N} \int_{B(0, R)} \left( \sum_{\substack{z, w \in (E_{-+}^\delta)^{-1}(0) \\ z \neq w}} \varphi(z, w) \right) e^{-\alpha^* \cdot \alpha} L(d\alpha) = \int_{\mathbb{C}^2} \varphi(z_1, z_2) d\nu(z_1, z_2)$$

with  $\varphi \in \mathcal{C}_0(\Omega \times \Omega)$ . Using Remark 14, we see that the integral is finite since the number of pairs of zeros of  $E_{-+}^\delta(\cdot, \alpha)$  in  $\text{supp } \varphi$  is uniformly bounded for  $\alpha \in B(0, R)$ .

Recall the definition of the point process  $\Xi$  given in (2.4). Using Lemma 7.1 in [28], we get the following regularization of the 2-fold counting measure  $\Xi \otimes \Xi$

$$\langle \varphi, \Xi \otimes \Xi \rangle = \lim_{\varepsilon \rightarrow 0^+} \iint \varphi(z_1, z_2) \prod_{j=1}^2 \varepsilon^{-2} \chi\left(\frac{E_{-+}^\delta(z_j)}{\varepsilon}\right) |\partial_{z_j} E_{-+}^\delta(z_j)|^2 L(dz_1) L(dz_2),$$

where  $\chi \in \mathcal{C}_0^\infty(\mathbb{C})$  such that  $\int \chi(w)L(dw) = 1$ . Assuming that  $\varphi \in \mathcal{C}_0(\Omega \times \Omega)$  is such that  $\{(z, z); z \in \Omega\} \cap \text{supp } \varphi = \emptyset$ , we see by the Lebesgue dominated convergence theorem that the two-point intensity measure of the point process  $\Xi$  is given by

$$\int_{\mathbb{C}^2} \varphi(z_1, z_2) d\nu(z_1, z_2) = \lim_{\varepsilon \rightarrow 0^+} \iint \varphi(z_1, z_2) K_\varepsilon^\delta(z_1, z_2; h) L(dz_1) L(dz_2) \quad (3.10)$$

with

$$K_\varepsilon^\delta(z_1, z_2; h) := \int_{B(0, R)} \left[ \prod_{l=1}^2 \varepsilon^{-2} \chi\left(\frac{E_{-+}^\delta(z_l)}{\varepsilon}\right) |\partial_{z_l} E_{-+}^\delta(z_l)|^2 \right] e^{-\alpha^* \alpha} L(d\alpha).$$

Using (3.8), we see that the main object of interest, encoding all the information needed for (3.10), is the random vector

$$\begin{aligned} F^\delta(z, w, \alpha; h) &= \begin{pmatrix} E_{-+}^\delta(z) \\ E_{-+}^\delta(w) \\ (\partial_z E_{-+}^\delta)(z) \\ (\partial_z E_{-+}^\delta)(w) \end{pmatrix} \\ &= \begin{pmatrix} E_{-+}(z) \\ E_{-+}(w) \\ (\partial_z E_{-+})(z) \\ (\partial_z E_{-+})(w) \end{pmatrix} - \delta \begin{pmatrix} X(z) \cdot \alpha \\ X(w) \cdot \alpha \\ (\partial_z X)(z) \cdot \alpha \\ (\partial_z X)(w) \cdot \alpha \end{pmatrix} + \begin{pmatrix} T(z, \alpha) \\ T(w, \alpha) \\ (\partial_z T)(z, \alpha) \\ (\partial_z T)(w, \alpha) \end{pmatrix}, \end{aligned} \quad (3.11)$$

where  $X(z)$ ,  $X(w)$  are given in Definition 12. It will be very useful in the sequel to define the following  $G$ .

$$G := \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \in \mathbb{C}^{4 \times 4}, \quad (3.12)$$

with

$$\begin{aligned} A &:= \begin{pmatrix} (X(z)|X(z)) & (X(z)|X(w)) \\ (X(w)|X(z)) & (X(w)|X(w)) \end{pmatrix}, \\ B &:= \begin{pmatrix} (X(z)|\partial_z X(z)) & (X(z)|\partial_w X(w)) \\ (X(w)|\partial_z X(z)) & (X(w)|\partial_w X(w)) \end{pmatrix}, \\ C &:= \begin{pmatrix} (\partial_z X(z)|\partial_z X(z)) & (\partial_z X(z)|\partial_w X(w)) \\ (\partial_w X(w)|\partial_z X(z)) & (\partial_w X(w)|\partial_w X(w)) \end{pmatrix}. \end{aligned} \quad (3.13)$$

Notice that the matrices  $A, B, C$  depend on  $h$ ; see Definition 12. Next, we will state a formula for the Lebesgue density of the two-point intensity measure  $\nu$  in terms of the permanent of the Shur complement of  $G$ , i.e.

$$\Gamma := C - B^* A^{-1} B. \quad (3.14)$$

The permanent of a matrix is defined as follows (cf. [15]):

**Definition 15.** Let  $(M_{ij})_{ij} = M \in \mathbb{C}^{n \times n}$  be a square matrix and let  $S_n$  denote the symmetric group of order  $n$ . The permanent of  $M$  is defined by

$$\text{perm } M := \sum_{\sigma \in S_n} \prod_{i=1}^n M_{i\sigma(i)}. \quad (3.15)$$

**Remark 16.** Although the definition of the permanent resembles closely to that of the determinant, the two objects are quite different. Many properties known to hold true for determinants, fail to be true for permanents. For our purposes it is enough to note that it is multi-linear and symmetric. For more details concerning permanents and their properties we refer the reader to [15].

We will prove the following result:

**Proposition 17.** Let  $\Omega \Subset \Sigma$  be as in Hypothesis 1. Let  $\delta > 0$  be as in Hypothesis 2 and let  $\Gamma$  be as in (3.14). Moreover, let  $D(\Omega, C_2)$  be as in (2.2). Then, there exists a smooth function

$$D^\delta(z, w; h) = \frac{\text{perm } \Gamma(z, w; h) + \mathcal{O}\left(e^{-\frac{1}{Ch}} + \delta h^{-\frac{52}{10}}\right)}{\pi^2 \left(\sqrt{\det A(z, w; h)} + \mathcal{O}\left(\delta h^{-\frac{3}{2}}\right)\right)^2} + \mathcal{O}\left(e^{-\frac{D}{h^2}}\right).$$

and there exists a constant  $C_2 > 0$  such that for all  $\varphi \in \mathcal{C}_0(\Omega^2 \setminus D_h(\Omega, C_2))$

$$\int_{\mathbb{C}^2} \varphi(z, w) d\nu(z, w) = \int_{\mathbb{C}^2} \varphi(z, w) D(z, w, h, \delta) L(d(z, w)).$$

**Remark 18.** The proof of Proposition 17 will take up most of the rest of this paper. Therefore we give a short overview on how we will proceed:

In Section 4, we give a formula for the scalar product  $(X(z)|X(w))$  by constructing holomorphic quasimodes for the operators  $(P_h - z)$  and  $(P_h - z)^*$  to approximate the eigenfunction  $e_0$  and  $f_0$ , and by using the method of stationary phase.

In Section 5, we will use this formula to study the invertibility of the matrices  $G, A$  and  $\Gamma$ . Furthermore, we will study the permanent of  $\Gamma$ .

In Section 6, we give a proof of Proposition 17.

#### 4. STATIONARY PHASE

In this section we are interested in the scalar product  $(X(z)|X(w))$ . Recall from Definition 12 that the vector  $X(z)$ ,  $z \in \Omega$ , is given by  $X_{j,k} = \widehat{e}_0(z; k) \widehat{f}_0(z; j)$ , where  $e_0$  and  $f_0$  are the eigenfunctions of the operators  $Q(z)$  and  $\tilde{Q}(z)$ , respectively, associated to their first eigenvalue  $t_0^2$ .

The Fourier coefficients  $\widehat{e}_0(z; k)$ ,  $\widehat{f}_0(z; j)$  and their  $z$ - and  $\bar{z}$ -derivatives are of order  $\mathcal{O}(|k|^{-\infty})$ ,  $\mathcal{O}(|j|^{-\infty})$ , for  $|j|, |k| \geq C/h$  with  $C > 0$  large enough (cf [28, Propositions 5.3 and 5.4]). The Parseval identity implies that for  $z, w \in \Omega$

$$(X(z)|X(w)) = (e_0(z)|e_0(w))(f_0(w)|f_0(z)) + \mathcal{O}_{\mathcal{C}^\infty}(h^\infty). \quad (4.1)$$

The aim of this section is to prove the following result:

**Proposition 19.** Let  $\Omega \Subset \Sigma$  be as in Hypothesis 1 and let  $x_\pm(z)$  be as in (1.5). Furthermore, for  $z \in \Omega$  let  $\sigma(z)$  denote the Lebesgue density of the direct image of the symplectic volume form on  $T^*S^1$  under the principal symbol  $p$ , i.e.  $\sigma(z)L(dz) = p_*(d\xi \wedge dx)$ .

Then, there exists a constant  $C > 0$  such that for all  $(z, w) \in \Delta_\Omega(C) := \{(z, w) \in \Omega^2; |z - w| < 1/C\}$

$$(X(z)|X(w)) = e^{-\frac{1}{h}\Phi(z; h) - \frac{1}{h}\Phi(w; h)} e^{\frac{2}{h}\Psi(z, w; h)} + \mathcal{O}_{\mathcal{C}^\infty}(h^\infty)$$



where:

- $\Phi(\cdot; h) : \Omega \rightarrow \mathbb{R}$  is a family of smooth functions depending only on  $i\text{Im } z$ , which satisfy

$$\begin{aligned} \Phi(z; h) = & \text{Im} \int_{x_+(z)}^{x_0} (z - g(y)) dy - \text{Im} \int_{x_-(z)}^{y_0} (z - g(y)) dy \\ & + \frac{h}{4} \left[ \ln \left( \frac{\pi h}{-\text{Im } g'(x_+(z))} \right) + \ln \left( \frac{\pi h}{\text{Im } g'(x_-(z))} \right) \right] + \mathcal{O}(h^2). \end{aligned}$$

and

$$\partial_{z\bar{z}}^2 \Phi(z; h) = \frac{1}{4} \sigma(z) + \mathcal{O}(h).$$

- $\Psi(\cdot, \cdot; h) : \Delta_\Omega(C) \rightarrow \mathbb{C}$  is a family of smooth functions which are almost  $z$ -holomorphic and almost  $w$ -anti-holomorphic extensions from the diagonal  $\Delta := \{(z, z); z \in \Omega\} \subset \Delta_\Omega(C)$  of  $\Phi(z; h)$ , i.e.

$$\Psi(z, z; h) = \Phi\left(\frac{1}{2}(z - \bar{z}); h\right), \quad \partial_{\bar{z}} \Psi, \partial_w \Psi = \mathcal{O}(|z - w|^\infty).$$

Moreover, we have that  $\Psi(z, z) = \Phi(z)$  and for  $z, w \in \Delta_\Omega(C)$  with  $|z - w| \ll 1$ ,

$$\begin{aligned} \Psi(z, w; h) = & \sum_{|\alpha+\beta| \leq 2} \frac{1}{2^{|\alpha+\beta|} \alpha! \beta!} \partial_z^\alpha \partial_{\bar{z}}^\beta \Phi\left(\frac{z+w}{2}; h\right) (z-w)^\alpha (\overline{w-z})^\beta \\ & + \mathcal{O}(|z-w|^3 + h^\infty), \end{aligned}$$

and

$$\begin{aligned} & 2\text{Re } \Psi(z, w; h) - \Phi(z; h) - \Phi(w; h) \\ & = -\partial_{z\bar{z}}^2 \Phi\left(\frac{z+w}{2}; h\right) |z-w|^2 (1 + \mathcal{O}(|z-w| + h^\infty)); \end{aligned}$$

- the function  $\Psi(z, w; h)$  has the following symmetries:

$$\Psi(z, w; h) = \overline{\Psi(w, z; h)} \quad \text{and} \quad (\partial_z \Psi)(z, w; h) = \overline{(\partial_{\bar{w}} \Psi)(w, z; h)}.$$

Let us give some remarks on the above results: Note that the formula for  $\Psi$  stated above is simply a special case of the more general Taylor expansion

$$\Psi(z_0 + \zeta, z_0 + \omega; h) = \sum_{|\alpha+\beta| \leq 2} \frac{1}{\alpha! \beta!} \partial_z^\alpha \partial_{\bar{z}}^\beta \Phi(z_0; h) \zeta^\alpha \bar{\omega}^\beta + \mathcal{O}((\zeta, \omega)^3 + h^\infty),$$

with  $z_0 \in \Omega$  and  $|\zeta|, |\omega| \ll 1$ .

To prove Proposition 19, we will use (4.1) and study the scalar products  $(e_0(z)|e_0(w))$  and  $(f_0(w)|f_0(z))$  separately, see Sections 4.1 and 4.2 below. The proof of Proposition 19 will then be stated at the end of this section.

**Remark 20.** Note that the behaviour of  $(X(z)|X(w))$  is close to the behaviour of Bergman kernels (see for example [30, Sec. 13.3]). However, we will not use this notion in the sequel.

Next, we define for  $(z, w) \in \Delta_\Omega(C)$ , as in Proposition 19,

$$\begin{aligned} -K(z, w) &:= 2\operatorname{Re} \Psi(z, w; h) - \Phi(z; h) - \Phi(w; h) \\ &= -\left(\sigma\left(\frac{z+w}{2}\right) + \mathcal{O}(h)\right) \frac{|z-w|^2}{4} (1 + \mathcal{O}(|z-w| + h^\infty)). \end{aligned} \quad (4.2)$$

From the above Proposition we can immediately deduce some growth properties of certain quantities that will become important in the sequel.

**Corollary 21.** *Under the assumptions of Proposition 19, we have that*

$$\begin{aligned} \bullet \quad & |(X(z)|X(w))| = e^{-\frac{K(z,w)}{h}} + \mathcal{O}_{\mathcal{C}^\infty}(h^\infty); \\ \bullet \quad & \|X(z)\|^2 \|X(w)\|^2 \pm |(X(z)|X(w))|^2 = \left(1 \pm e^{-\frac{2K(z,w)}{h}}\right) + \mathcal{O}_{\mathcal{C}^\infty}(h^\infty); \\ \bullet \quad & \|X(z)\|^2 \|X(w)\|^2 |(X(z)|X(w))|^2 = e^{-\frac{2K(z,w)}{h}} + \mathcal{O}_{\mathcal{C}^\infty}(h^\infty). \end{aligned} \quad (4.3)$$

**4.1. The Scalar Product**  $(e_0(z)|e_0(w))$ . We will prove

**Proposition 22.** *Let  $\Omega \Subset \Sigma$  be as in Hypothesis 1 and let  $x_+(z)$  be as in (1.5). Then, there exists a constant  $C > 0$  such that for all  $(z, w) \in \Delta_\Omega(C) := \{(z, w) \in \Omega^2; |z-w| < 1/C\}$*

$$(e_0(z)|e_0(w)) = e^{-\frac{1}{h}\Phi_1(z;h)} e^{-\frac{1}{h}\Phi_1(w;h)} e^{\frac{2}{h}\Psi_1(z,w;h)} + \mathcal{O}(h^\infty), \quad (4.4)$$

where:

- $\Phi_1(\cdot; h) : \Omega \rightarrow \mathbb{R}$  is a family of smooth functions depending only on  $i\operatorname{Im} z$ , which satisfy

$$\Phi_1(z; h) = \operatorname{Im} \int_{x_+(\operatorname{Im} z)}^{x_0} (z - g(y)) dy + \frac{h}{4} \ln \left( \frac{\pi h}{-\operatorname{Im} g'(x_+)} \right) + \mathcal{O}(h^2).$$

- $\Psi_1(\cdot, \cdot; h) : \Delta_\Omega(C) \rightarrow \mathbb{C}$  is a family of smooth functions which are almost  $z$ -holomorphic and almost  $w$ -anti-holomorphic extensions from the diagonal  $\Delta := \{(z, z); z \in \Omega\} \subset \Delta_\Omega(C)$  of  $\Phi_1(z; h)$ , i.e.

$$\Psi_1(z, z; h) = \Phi_1\left(\frac{1}{2}(z - \bar{z}); h\right), \quad \partial_{\bar{z}}\Psi_1, \partial_w\Psi_1 = \mathcal{O}(|z-w|^\infty).$$

Moreover, for  $z, w \in \Delta_\Omega(C)$  with  $|z-w| \ll 1$ , one has that

$$\begin{aligned} \Psi_1(z, w; h) &= \sum_{|\alpha+\beta| \leq 2} \frac{1}{2^{|\alpha+\beta|} \alpha! \beta!} \partial_z^\alpha \partial_{\bar{z}}^\beta \Phi_1\left(\frac{z+w}{2}; h\right) (z-w)^\alpha (\overline{w-z})^\beta \\ &\quad + \mathcal{O}(|z-w|^3 + h^\infty), \end{aligned}$$

and that

$$\begin{aligned} & 2\operatorname{Re} \Psi_1(z, w; h) - \Phi_1(z; h) - \Phi_1(w; h) \\ &= -\partial_z \partial_{\bar{z}} \Phi_1\left(\frac{z+w}{2}; h\right) |z-w|^2 (1 + \mathcal{O}(|z-w| + h^\infty)); \end{aligned}$$

- the function  $\Psi_1(z, w; h)$  has the following symmetries:

$$\Psi_1(z, w; h) = \overline{\Psi_1(w, z; h)} \quad \text{and} \quad (\partial_z \Psi_1)(z, w; h) = \overline{(\partial_{\bar{w}} \Psi_1)(w, z; h)}.$$

To prove Proposition 22, we begin by constructing an oscillating function to approximate  $e_0(z)$ . Let us recall from Section 1 that the points  $a, b \in S^1$  denote the minimum and the maximum of  $\operatorname{Im} g(x)$  and that for  $z \in \Omega$  the points  $x_{\pm}(z) \in S^1$  are the unique solutions to the equation  $\operatorname{Im} g(x) = \operatorname{Im} z$ . Furthermore, we will identify frequently  $S^1$  with the interval  $[b - 2\pi, b[$ . Moreover, let us recall that by the natural projection  $\Pi : \mathbb{R} \rightarrow S^1 = \mathbb{R}/2\pi\mathbb{Z}$  we identify the points  $x_{\pm}, a, b \in S^1$  with points  $x_{\pm}, a, b \in \mathbb{R}$  such that  $b - 2\pi < x_+ < a < x_- < b$ .

Let  $K_+ \subset ]b - 2\pi, a[$  be an open interval such that  $x_+(z) \in K_+$  for all  $z \in \Omega$ . Let  $\chi \in \mathcal{C}_0^\infty([b - 2\pi, a])$  and define for  $x \in \mathbb{R}$

$$\tilde{e}_0(x, z) := \chi(x) \exp\left(\frac{i}{h}\psi_+(x, z)\right). \quad (4.5)$$

where, for a fixed  $x_0 \in K_+$ ,

$$\psi_+(x, z) := \int_{x_0}^x (z - g(y)) dy. \quad (4.6)$$

**Remark 23.** Note that the function  $u = \exp(i\psi_+(x, z)/h)$  is solution to  $(P_h - z)u = 0$  on  $\operatorname{supp} \chi$ , since the phase function  $\psi_+$  satisfies the eikonal equation

$$p(x, \partial_x \psi_+) = z.$$

Furthermore, let us remark that  $\tilde{e}_0(x, z)$  depends holomorphically on  $z$ .

Next, we are interested in the  $L^2$ -norm of  $\tilde{e}_0$ .

**Lemma 24.** Let  $\Omega \Subset \Sigma$  be as in Hypothesis 1. Then, there exists a family of smooth functions  $\Phi_1(\cdot; h) : \Omega \rightarrow \mathbb{R}$ , such that

$$\Phi_1(z; h) = \Phi_1(i\operatorname{Im} z; h) = \operatorname{Im} \int_{x_+(\operatorname{Im} z)}^{x_0} (z - g(y)) dy + \frac{h}{4} \ln \left( \frac{\pi h}{-\operatorname{Im} g'(x_+)} \right) + \mathcal{O}(h^2)$$

and

$$\|\tilde{e}_0(z)\|^2 = \exp \left\{ \frac{2}{h} \Phi_1(z; h) \right\}.$$

*Proof.* In view of the definition of  $\tilde{e}_0(z)$ , see (4.5) and (4.6), one gets that

$$\|\tilde{e}_0(z)\|^2 = \int \chi(x) e^{\frac{i}{h}(\psi_+(x, z) - \bar{\psi}_+(x, z))} dx = \int \chi(x) e^{-\frac{2}{h} \operatorname{Im} \psi_+(x, z)} dx.$$

The critical point for  $\operatorname{Im} \psi_+(x, z)$  is given by the equation

$$\operatorname{Im} \partial_x \psi_+(x, z) = \operatorname{Im} z - \operatorname{Im} g(x) = 0, \quad x \in \operatorname{supp} \chi.$$

The critical point, given by  $x_+(\operatorname{Im} z)$ , is unique and it satisfies  $\operatorname{Im} g'(x_+(\operatorname{Im} z)) < 0$ , see (1.5). This implies in particular that the critical point is non-degenerate. More precisely,

$$\operatorname{Im} (\partial_{xx}^2 \psi_+)(x_+, z) = -\operatorname{Im} g'(x_+) > 0. \quad (4.7)$$

The critical value of  $\operatorname{Im} \psi_+$  is given by

$$\operatorname{Im} \psi_+(x_+(\operatorname{Im} z), z) = \operatorname{Im} \int_{x_0}^{x_+(\operatorname{Im} z)} (z - g(y)) dy \leq 0.$$

Using the method of stationary phase, one gets

$$\begin{aligned}\|\tilde{e}_0(z)\|^2 &= \sqrt{\frac{\pi h}{\operatorname{Im}(\partial_{xx}^2 \psi_+)(x_+, z)}} (1 + \mathcal{O}(h)) \exp\left\{-\frac{2\operatorname{Im} \psi_+(x_+, z)}{h}\right\} \\ &=: \exp\left\{\frac{2}{h}\Phi_1(z; h)\right\},\end{aligned}$$

where  $\Phi_1$  is smooth in  $z$ . Using (4.7), one gets that

$$\Phi_1(z; h) = \operatorname{Im} \int_{x_+(\operatorname{Im} z)}^{x_0} (z - g(y)) dy + \frac{h}{4} \ln \left( \frac{\pi h}{-\operatorname{Im} g'(x_+)} \right) + \mathcal{O}(h^2). \quad \square$$

Recall from (3.3) that the function  $e_0$  is an eigenfunction of the operator  $Q(z)$  (cf Section 3.2) corresponding to its first eigenvalue  $t_0^2$ . We set

$$e_0(z) = \frac{\Pi_{t_0^2} \left( e^{-\frac{1}{h}\Phi_1(z; h)} \tilde{e}_0(z) \right)}{\left\| \Pi_{t_0^2} \left( e^{-\frac{1}{h}\Phi_1(z; h)} \tilde{e}_0(z) \right) \right\|},$$

where  $\Pi_{t_0^2} : L^2(S^1) \rightarrow \mathbb{C}e_0$  denotes the spectral projection for  $Q(z)$  onto the eigenspace associated with  $t_0^2$ .

Next, we prove that up to an exponentially small error in  $1/h$ ,  $e_0$  is given by the normalization of  $\tilde{e}_0$ .

**Lemma 25.** *Let  $\Omega \Subset \Sigma$  be as in Hypothesis 1. Then, there exists a constant  $C > 0$  such that for all  $z \in \Omega$  and all  $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$*

$$\left\| \partial_z^{\alpha_1} \partial_{\bar{z}}^{\alpha_2} \left( e_0(z) - e^{-\frac{1}{h}\Phi_1(z; h)} \tilde{e}_0(z) \right) \right\| = \mathcal{O}\left(h^{-|\alpha|} e^{-\frac{1}{Ch}}\right).$$

*Proof.* The proof of the lemma is similar to the proof of [28, Proposition 3.11].  $\square$

This result implies that

$$(e_0(z)|e_0(w)) = e^{-\frac{1}{h}\Phi_1(z; h) - \frac{1}{h}\Phi_1(w; h)} (\tilde{e}_0(z)|\tilde{e}_0(w)) + \mathcal{O}_{C^\infty}\left(e^{-\frac{1}{Ch}}\right). \quad (4.8)$$

By Remark 23,  $(\tilde{e}_0(z)|\tilde{e}_0(w))$  is holomorphic in  $z$  and anti-holomorphic in  $w$ . We can study this scalar product by the method of stationary phase:

of Proposition 22. In view of (4.8), it remains to study the oscillatory integral

$$I(z, w) := (\tilde{e}_0(z)|\tilde{e}_0(w)) = \int \chi(x) \exp\left(\frac{i}{h}\Psi_+(x, z, w)\right) dx, \quad (4.9)$$

where  $\tilde{e}_0(x, z)$  is given in (4.5) and  $\Psi_+$  is defined by

$$\Psi_+(x, z, w) := \psi_+(x, z) - \overline{\psi_+(x, w)}, \quad z, w \in \Omega. \quad (4.10)$$

Using (4.6),

$$\Psi_+(x, z, w) = \int_{x_0}^x \operatorname{Re}(z - w) dy + 2i \int_{x_0}^x \left[ \operatorname{Im}\left(\frac{z + w}{2}\right) - \operatorname{Im} g(y) \right] dy. \quad (4.11)$$

Since the imaginary part of  $\Psi_+$  can be negative, we shift the phase function by the minimum of  $\operatorname{Im} \Psi_+$ .

**Minimum of  $\text{Im } \Psi_+$ .** The critical points of the function  $x \mapsto \text{Im } \Psi(x, z, w)$  are given by the equation  $\text{Im} \left( \frac{z+w}{2} \right) = \text{Im } g(x)$ . Since  $\Omega$  is convex, this equation has, for  $|z - w|$  small enough, on the support of  $\chi$  the unique solution  $x_+(\frac{z+w}{2}) \in \mathbb{R}$  and it satisfies  $\text{Im } g'(x_+(\frac{z+w}{2})) < 0$  (cf. (1.5)). Moreover, it depends smoothly on  $z$  and  $w$  since  $g$  is smooth. Therefore,

$$(\partial_{xx}^2 \text{Im } \Psi_+) \left( x_+ \left( \frac{z+w}{2} \right), z, w \right) = -2 \text{Im } g'_x \left( x_+ \left( \frac{z+w}{2} \right) \right) > 0,$$

which implies that  $x_+(\frac{z+w}{2})$  is a minimum point, and that

$$\begin{aligned} 2\lambda := 2\lambda(z, w) &:= \text{Im } \Psi_+ \left( x_+ \left( \frac{z+w}{2} \right), z, w \right) \\ &= 2 \int_{x_0}^{x_+(\frac{z+w}{2})} \left[ \text{Im} \left( \frac{z+w}{2} \right) - \text{Im } g(y) \right] dy \leq 0. \end{aligned} \quad (4.12)$$

We define  $\Theta_+(x, z, w) := \Psi_+(x, z, w) - i\lambda$ , and notice that  $\text{Im } \Theta_+(x, z, w) \geq 0$ . Hence, we can write (4.9) as follows:

$$I(z, w) = e^{-\frac{2\lambda}{h}} \int \chi(x) \exp \left( \frac{i}{h} \Theta_+(x, z, w) \right) dx. \quad (4.13)$$

To study  $I(z, w)$  by the method of stationary phase, we are interested in the critical points of  $\Theta_+$ .

**Critical points of  $\Theta_+$ .** Clearly they are the same as for  $\Psi_+(x, z, w)$ . Note that for  $z = w$  one has that

$$\Psi_+(x, z, z) = 2i \text{Im} \int_{x_0}^x (z - g(y)) dy$$

which has, on the support of  $\chi$ , the unique critical point  $x_+$  and it satisfies  $\text{Im } g'(x_+) < 0$  (cf. (1.5)). Therefore,

$$\text{Im} (\partial_{xx}^2 \Psi_+)(x_+(z), z, z) = -2 \text{Im } g'_x(x_+(z)) > 0$$

which implies that  $x_+$  is a non-degenerate critical point.

In the case where  $z \neq w$  the situation is more complicated. By (4.11) we see that if  $\text{Re}(z - w) = 0$ , for  $|z - w|$  small enough, the critical point is real and given by  $x_+(\frac{z+w}{2})$ , i.e. the minimum point of  $\text{Im } \Psi_+$ .

However, if  $\text{Re}(z - w) \neq 0$ , we need to consider an almost  $x$ -analytic extension of  $\Psi_+$ , which we shall denote by  $\tilde{\Psi}_+$ . As described in [16], the “critical point” of  $\tilde{\Psi}_+$  is then given by

$$\partial_x \tilde{\Psi}_+(x, z, w) = 0,$$

and we will see, by the following result, that it “moves” to the complex plane.

**Lemma 26.** *Let  $\Omega \Subset \Sigma$  be as in (1.9). Let  $\chi$  be as in (4.5) and let  $p$  be the principal symbol of  $P_h$  (cf (1.3)). Let  $x_+(z)$  be as in (1.5). Furthermore, let  $\tilde{\psi}_+$  denote an almost analytic extension of  $\psi_+$  to a small complex neighborhood of the support of  $\chi$ , and define*

$\tilde{\psi}_+^*(x) := \overline{\tilde{\psi}_+(x)}$ . Then, there exists a  $C > 0$  such that for  $(z, w) \in \Delta_\Omega(C)$  the function

$$\partial_x \tilde{\Psi}_+(x, z, w) = \partial_x \tilde{\psi}_+(x, z) - (\partial_x \tilde{\psi}_+)^*(x, w)$$

has exactly one zero,  $x_+^c(z, w)$ , and:

- it depends almost holomorphically on  $z$  and almost anti-holomorphically on  $w$  at the diagonal  $\Delta$ , i.e.

$$\partial_w x_+^c(z, w), \partial_{\bar{z}} x_+^c(z, w) = \mathcal{O}(|z - w|^\infty);$$

- it is non-degenerate in the sense that

$$(\partial_{xx}^2 \tilde{\Psi}_+)(x_+^c(z, w), z, w) \neq 0;$$

- for  $z, w \in \Omega$  with  $|z - w| < 1/C$ ,  $C > 1$  large enough, one has

$$x_+^c(z, w) = x_+ \left( \frac{z + w}{2} \right) - \frac{\operatorname{Re}(z - w)}{\{p, \bar{p}\}(\rho_+(\frac{z+w}{2}))} + \mathcal{O}(|z - w|^2).$$

**Remark 27.** The proof of Lemma 26 will be given after the proof of Proposition 22.

Let  $\tilde{\Psi}_+$  denote an almost  $x$ -analytic extension of  $\Psi_+$ . Using the method of stationary phase for complex-valued phase functions (cf. Theorem 2.3 in [16, p.148]) and Lemma 26, one gets that

$$I(z, w) = \exp \left\{ \frac{2\Psi_1(z, w; h)}{h} \right\} + \mathcal{O}(h^\infty) e^{-\frac{2\lambda}{h}}. \quad (4.14)$$

Using that Lemma 24 and (4.12) imply  $\lambda(z, w) + \Phi(z; h) + \Phi(w; h) \geq 0$ , we obtain (4.4) from the above and (4.8).

In (4.14),  $2\Psi_1(z, w)$  is given by the critical value of  $i\tilde{\Psi}_+$  and by the logarithm of the amplitude  $c(z, w, h)$ , given by the stationary phase method, i.e.

$$2\Psi_1(z, w; h) = i\tilde{\Psi}_+(x_+^c(z, w), z, w) + h \ln c(z, w, h)$$

and  $c(z, w, h) \sim c_0(z, w) + hc_1(z, w) + \dots$  which depends smoothly on  $z$  and  $w$  in the sense that all  $z$ -,  $\bar{z}$ -,  $w$ - and  $\bar{w}$ -derivatives remain bounded as  $h \rightarrow 0$ .  $\tilde{\Psi}_+(x, z, w)$  is by definition  $z$ -holomorphic,  $w$ -anti-holomorphic and smooth in  $x$ . By Lemma 26, we know that the critical point  $x_+^c(z, w)$  is almost  $z$ -holomorphic and almost  $w$ -anti-holomorphic in  $\Delta_\Omega(C)$ , a small neighborhood of the diagonal  $z = w$ . Hence,  $\Psi$  is almost  $z$ -holomorphic and almost  $w$ -anti-holomorphic in  $\Delta_\Omega(C)$ .

Equivalently,  $\Psi$  is an almost  $z$ -holomorphic and almost  $w$ -anti-holomorphic extension from the diagonal of  $\Psi_1(z, z; h)$ . Since  $\Psi_1(z, z; h) = \Phi_1(z; h)$ , we obtain by Taylor expansion up to order 2 of  $\Psi$  at  $(\frac{z+w}{2}, \frac{z+w}{2})$ , that

$$\begin{aligned} \Psi_1(z, w; h) &= \sum_{|\alpha+\beta| \leq 2} \frac{1}{2^{|\alpha+\beta|} \alpha! \beta!} \partial_z^\alpha \partial_{\bar{z}}^\beta \Phi_1 \left( \frac{z+w}{2}; h \right) (z-w)^\alpha (\overline{w-z})^\beta \\ &\quad + \mathcal{O}(|z-w|^3 + h^\infty), \end{aligned}$$

for  $|z - w|$  small enough. Similarly,

$$\begin{aligned} \Phi_1(z; h) &= \sum_{|\alpha+\beta| \leq 2} \frac{1}{2^{|\alpha+\beta|} \alpha! \beta!} \partial_z^\alpha \partial_{\bar{z}}^\beta \Phi_1 \left( \frac{z+w}{2}; h \right) (z-w)^\alpha (\overline{z-w})^\beta \\ &\quad + \mathcal{O}(|z-w|^3 + h^\infty), \end{aligned}$$

which implies that

$$\begin{aligned} 2\operatorname{Re} \Psi_1(z, w; h) &= \Phi_1(z; h) + \Phi_1(w; h) - \partial_z^\alpha \partial_{\bar{z}}^\beta \Phi_1 \left( \frac{z+w}{2}; h \right) |z-w|^2 \\ &\quad + \mathcal{O}(|z-w|^3 + h^\infty), \end{aligned}$$

concluding the proof of the second point of the proposition.

Finally, let us give a proof of the stated symmetries. The fact that  $\Psi_1(z, w; h) = \overline{\Psi_1(w, z; h)}$  follows directly from the fact that  $(e_0(z)|e_0(w)) = \overline{(e_0(w)|e_0(z))}$ . One then computes that

$$(\partial_z \Psi_1)(z, w; h) = \partial_z \Psi_1(z, w; h) = \overline{\partial_{\bar{z}} \Psi_1(w, z; h)} = \overline{(\partial_{\bar{w}} \Psi_1)(w, z; h)}$$

which concludes the proof of the Proposition.  $\square$

of Lemma 26. We are interested in the solutions of the following equation:

$$0 = (\partial_x \tilde{\psi}_+)(x, z) - (\partial_x \tilde{\psi}_+)^*(x, w) = z - \bar{w} - \tilde{g}(x) + \tilde{g}^*(x), \quad (4.15)$$

where  $\tilde{g}$  denotes an almost analytic extension of  $g$ . Since  $\operatorname{dist}(\Omega, \partial\Sigma) > 1/C$ , it follows from the assumptions on  $g$  that  $\operatorname{Im} g'(x) > 0$  for all  $x \in x_+(\Omega) \subset \mathbb{R}$ . Since  $g$  depends smoothly on  $x$ , there exists a small complex open neighborhood  $V \subset \mathbb{C}$  of  $x_+(\Omega)$  such that  $\overline{x_+(\Omega)} \subset (V \cap \mathbb{R})$  and such that for all  $x \in V$

$$\tilde{g}'_x(x) - \overline{\tilde{g}'_x(\bar{x})} \neq 0, \quad \tilde{g}'_{\bar{x}}(x) - \overline{\tilde{g}'_{\bar{x}}(\bar{x})} = \mathcal{O}(|\operatorname{Im} x|^\infty).$$

Thus, it follows by the implicit function theorem, that for  $(z, w) \in \Delta_\Omega(C)$ , with  $C > 0$  large enough, there exists a unique solution  $x_+^c(z, w)$  to (4.15) and it depends smoothly on  $(z, w) \in \Delta_\Omega(C)$ . Furthermore, we have that  $x_+^c(z, z) = x_+(z) \in \mathbb{R}$ . Taking the  $z$ - and  $\bar{z}$ - derivative of (4.15) at the critical point  $x_+^c$  yields that

$$\begin{aligned} \partial_z x_+^c(z, w) &= \frac{1 + \mathcal{O}(|\operatorname{Im} x_+^c(z, w)|^\infty)}{(\partial_x \tilde{g})(x_+^c(z, w)) - (\partial_x \tilde{g})^*(x_+^c(z, w))}, \\ \partial_{\bar{z}} x_+^c(z, w) &= \frac{\mathcal{O}(|\operatorname{Im} x_+^c(z, w)|^\infty)}{(\partial_x \tilde{g})(x_+^c(z, w)) - (\partial_x \tilde{g})^*(x_+^c(z, w))} \end{aligned} \quad (4.16)$$

and similarly that

$$\begin{aligned} \partial_{\bar{w}} x_+^c(z, w) &= \frac{-1 + \mathcal{O}(|\operatorname{Im} x_+^c(z, w)|^\infty)}{(\partial_x \tilde{g})(x_+^c(z, w)) - (\partial_x \tilde{g})^*(x_+^c(z, w))}, \\ \partial_w x_+^c(z, w) &= \frac{\mathcal{O}(|\operatorname{Im} x_+^c(z, w)|^\infty)}{(\partial_x \tilde{g})(x_+^c(z, w)) - (\partial_x \tilde{g})^*(x_+^c(z, w))}. \end{aligned} \quad (4.17)$$

Using that  $\operatorname{Im} x_+^c(z, z) = 0$ , one calculates that for  $z = w$  we have that

$$\begin{aligned} (\partial_z x_+^c)(z, z) &= \partial_z x_+(z) = -(\partial_{\bar{w}} x_+^c)(z, z), \\ \text{and } (\partial_{\bar{z}} x_+^c)(z, z) &= 0 = (\partial_w x_+^c)(z, z), \end{aligned} \quad (4.18)$$

where

$$\partial_z x_+(z) = \frac{1}{2i \operatorname{Im} g'(x_+(z))}.$$

Taylor's theorem implies that

$$x_+^c(z + \zeta, z + \omega) = x_+(z) + \frac{\zeta - \bar{\omega}}{2i \operatorname{Im} g'(x_+(z))} + \mathcal{O}((\zeta, \omega)^2).$$

Recall that the principal symbol of the operator  $P_h$  is given by  $p(\rho) = \xi + g(x)$  (cf (1.3)), which implies that  $\{p, \bar{p}\}(\rho_{\pm}(z)) = -2i \operatorname{Im} g'(x_{\pm}(z))$ . To conclude the symmetric form of the Taylor expansion stated in the Lemma, we expand around the point  $(\frac{z+w}{2}, \frac{z+w}{2})$ , for  $|z - w|$  small enough, with  $\zeta = \frac{z-w}{2}$  and  $\omega = -\frac{z-w}{2}$ , which is possible since  $\Omega$  is by (1.9) assumed to be convex.

Finally, by taking the imaginary part of the Taylor expansion of  $x_+^c$ , we conclude by (4.16) and (4.17) that

$$\partial_w x_+^c(z, w), \partial_{\bar{z}} x_+^c(z, w) = \mathcal{O}(|z - w|^\infty). \quad \square$$

**4.2. The Scalar Product  $(f_0(w)|f_0(z))$ .** We have, as in Section 4.1,

**Proposition 28.** *Let  $\Omega \Subset \Sigma$  be as in Hypothesis 1 and let  $x_-(z)$  be as in (1.5). Then, there exists a constant  $C > 0$  such that for all  $(z, w) \in \Delta_\Omega(C) := \{(z, w) \in \Omega^2; |z - w| < 1/C\}$*

$$(f_0(w)|f_0(z)) = e^{-\frac{1}{h}\Phi_2(z;h)} e^{-\frac{1}{h}\Phi_2(w;h)} e^{\frac{2}{h}\Psi_2(z,w;h)} + \mathcal{O}(h^\infty),$$

where:

- $\Phi_2(\cdot; h) : \Omega \rightarrow \mathbb{R}$  is a family of smooth functions depending only on  $\operatorname{Im} z$ , which satisfy

$$\Phi_2(z; h) = -\operatorname{Im} \int_{x_-(z)}^{x_0} (z - g(y)) dy + \frac{h}{4} \ln \left( \frac{\pi h}{\operatorname{Im} g'(x_-(z))} \right) + \mathcal{O}(h^2).$$

- $\Psi_2(\cdot, \cdot; h) : \Delta_\Omega(C) \rightarrow \mathbb{C}$  is a family of smooth functions which are almost  $z$ -holomorphic and almost  $w$ -anti-holomorphic extensions from the diagonal  $\Delta := \{(z, z); z \in \Omega\} \subset \Delta_\Omega(C)$  of  $\Phi_2(z; h)$ , i.e.

$$\partial_{\bar{z}} \Psi_2, \partial_w \Psi_2 = \mathcal{O}(|z - w|^\infty), \quad \Psi_2(z, z; h) = \Phi_2 \left( \frac{1}{2}(z - \bar{z}); h \right)$$

Moreover, for  $z, w \in \Delta_\Omega(C)$  with  $|z - w| \ll 1$ , one has that

$$\begin{aligned} \Psi_2(z, w; h) &= \sum_{|\alpha+\beta| \leq 2} \frac{1}{2^{|\alpha+\beta|} \alpha! \beta!} \partial_z^\alpha \partial_{\bar{z}}^\beta \Phi_2 \left( \frac{z+w}{2}; h \right) (z-w)^\alpha (\overline{w-z})^\beta \\ &\quad + \mathcal{O}(|z - w|^3 + h^\infty), \end{aligned}$$



and that

$$\begin{aligned} 2\operatorname{Re} \Psi_2(z, w; h) - \Phi_2(z; h) - \Phi_2(w; h) \\ = -\partial_z \partial_{\bar{z}} \Phi_2 \left( \frac{z+w}{2}; h \right) |z-w|^2 (1 + \mathcal{O}(|z-w| + h^\infty)); \end{aligned}$$

- the function  $\Psi_2(z, w; h)$  has the following symmetries:

$$\Psi_2(z, w; h) = \overline{\Psi_2(w, z; h)} \quad \text{and} \quad (\partial_z \Psi_2)(z, w; h) = \overline{(\partial_{\bar{w}} \Psi_2)(w, z; h)}.$$

**4.3. Link with the symplectic volume.** Before the proof of Proposition 19, let us give a short description of the connection between the functions  $\Phi_1(z; h)$ ,  $\Phi_2(z; h)$  in Proposition 22, 28, and the symplectic volume form on the phase space  $T^*S^1$ .

**Proposition 29.** *Let  $z \in \Omega \Subset \Sigma$  be as in (1.9) and let  $\Phi_1$  and  $\Phi_2$  be as in Propositions 22 and 28. Furthermore, let  $p$  be the principal symbol of  $P_h$  (cf (1.3)), let  $\rho_\pm \in T^*S^1$  be the two solutions to  $p(\rho) = z$ , see (1.5). Then,*

$$\begin{aligned} \sigma_h(z) &:= [(\partial_{z\bar{z}}^2 \Phi_1)(z; h) + (\partial_{z\bar{z}}^2 \Phi_2)(z; h)] \\ &= \frac{1}{4} \left( \frac{1}{\frac{1}{2i}\{p, \bar{p}\}(\rho_-(z))} + \frac{1}{\frac{1}{2i}\{p, \bar{p}\}(\rho_+(z))} \right) + \mathcal{O}(h) \end{aligned}$$

is, up to an error of order  $h$ , one-fourth of the Lebesgue density of the direct image, under the principal symbol  $p$ , of the symplectic volume form  $d\xi \wedge dx$  on  $T^*S^1$ , i.e.

$$\sigma_h(z)L(dz) = \frac{1}{4}p_*(d\xi \wedge dx) + \mathcal{O}(h)L(dz)$$

*Proof.* Using that  $x_\pm(t)$ , with  $t = \operatorname{Im} z$ , is the solution to the equation  $\operatorname{Im} g(x_\pm(t)) = t$  with

$$\mp \operatorname{Im} g'_x(x_\pm(t)) < 0$$

(cf (1.5)), we get that

$$x'_\pm(t) = \pm \frac{1}{\operatorname{Im} g'_x(x_\pm(t))} < 0.$$

Using Propositions 22 and 28, one then computes that

$$(\partial_{z\bar{z}}^2 \Phi_1)(z; h) + (\partial_{z\bar{z}}^2 \Phi_2)(z; h) = \frac{1}{4} \left( \frac{1}{\operatorname{Im} g'_x(x_-(\operatorname{Im} z))} - \frac{1}{\operatorname{Im} g'_x(x_+(\operatorname{Im} z))} \right) + \mathcal{O}(h).$$

Since  $-\frac{1}{2i}\{p, \bar{p}\}(\rho_\pm) = \operatorname{Im} g'_x(x_\pm)$ , we conclude by Proposition 6.2 in [28] that

$$[(\partial_{z\bar{z}}^2 \Phi_1)(z; h) + (\partial_{z\bar{z}}^2 \Phi_2)(z; h)] L(dz) = \frac{1}{4}p_*(d\xi \wedge dx) + \mathcal{O}(h)L(dz). \quad \square$$

of Proposition 19. The results follow immediately from (4.1) and the Propositions 22, 28 and 29.  $\square$

## 5. GRAMIAN MATRIX

The aim of this section is to study the Gramian matrix  $G$  which is defined in (3.12) via the blocks  $A$ ,  $B$ , and  $C$ , given in (3.13). This will be essential to the proof of Proposition 17. Most of the results obtained here follow from involved but straightforward calculations which use strongly Proposition 19, the principal result of the previous section.

This section is organized as follows: in Section 5.1 we discuss the invertibility of the matrix  $A$  and provide estimates for its determinant. In Section 5.2 we obtain detailed formulas for  $\Gamma$ , which is given by the Shur complement formula applied to  $G$  (cf. (3.12), (3.14)), i.e.

$$\Gamma = C - B^* A^{-1} B. \quad (5.1)$$

In Section 5.3 we will discuss the invertibility of the matrix  $G$  and in Section 5.4 we will state a formula for the permanent of  $\Gamma$  which is an essential quantity of Proposition 17.

**5.1. The matrix  $A$ .** We begin by studying the determinant of  $A$ , cf. (3.13). It is non-zero if and only if the vectors  $X(z)$  and  $X(w)$  (given in Definition 12) are not co-linear. In particular we are interested in a lower bound of this determinant for  $z$  and  $w$  close.

**Proposition 30.** *Let  $\Omega \Subset \Sigma$  be as in Hypothesis 1 and let  $A$  be as in (3.13). For  $z, w \in \Omega$  with  $|z - w| \leq 1/C$ , with  $C > 1$  large enough (cf. Proposition 19), we have*

$$\det A(z, w) = 1 - e^{-\frac{2K(z, w)}{h}} + \mathcal{O}_{C^\infty}(h^\infty),$$

where  $K(z, w)$  is as in (4.2). Moreover,

- for  $|z - w| \gg \sqrt{h \ln h^{-1}}$

$$\det A(z, w) = 1 + \mathcal{O}(h^C), \quad C \gg 1;$$

- for  $|z - w| \geq \frac{1}{\mathcal{O}(1)} \sqrt{h}$

$$\det A \geq \frac{1}{\mathcal{O}(1)};$$

- let  $N > 1$  and let  $C > 1$  be large enough, then for  $\frac{1}{C} h^N \leq |z - w| \leq \frac{1}{C} \sqrt{h}$ ,

$$\begin{aligned} \det A(z, w) &= \frac{|z - w|^2}{2h} \left( \sigma \left( \frac{z + w}{2} \right) + \mathcal{O}(h) + \mathcal{O}(|z - w|) + \mathcal{O} \left( \frac{|z - w|^2}{h} \right) \right) \\ &\quad + \mathcal{O}_{C^\infty}(h^\infty) \\ &\geq \frac{h^{2N-1}}{\mathcal{O}(1)}. \end{aligned}$$

*Proof.* By Corollary 21 and (4.2), one has that

$$\det A(z, w) = 1 - e^{-\frac{2K(z, w)}{h}} + \mathcal{O}_{C^\infty}(h^\infty),$$

with

$$K(z, w) = \left( \sigma \left( \frac{z + w}{2} \right) + \mathcal{O}(h) \right) \frac{|z - w|^2}{4} (1 + \mathcal{O}(|z - w| + h^\infty)).$$

The first two estimates are then an immediate consequence of the above formula. In the case where  $|z - w| \leq \frac{1}{C}\sqrt{h}$ , one computes, using Taylor's formula, that

$$e^{-\frac{2K(z,w)}{h}} = 1 - \frac{|z - w|^2}{2h} \left( \sigma \left( \frac{z + w}{2} \right) + \mathcal{O}(h) + \mathcal{O}(|z - w|) + \mathcal{O} \left( \frac{|z - w|^2}{h} \right) \right),$$

which implies that

$$\begin{aligned} \det A(z, w) &= \frac{|z - w|^2}{2h} \left( \sigma \left( \frac{z + w}{2} \right) + \mathcal{O}(h) + \mathcal{O}(|z - w|) + \mathcal{O} \left( \frac{|z - w|^2}{h} \right) \right) + \mathcal{O}_{C^\infty}(h^\infty) \\ &\geq \frac{h^{2N-1}}{\mathcal{O}(1)}. \end{aligned}$$

□

Since the matrix  $A$  is self-adjoint, we have a lower bound on the matrix norm of  $A$  by its smallest eigenvalue. Using Proposition 19 we see that  $\text{tr } A = 2 + \mathcal{O}(h^\infty)$  and one calculates that for a fixed  $N > 1$  and for  $|z - w| \geq \frac{h^N}{\mathcal{O}(1)}$  the two eigenvalues of  $A$  are given by

$$\lambda_{1,2}(z, w; h) = 1 \pm e^{-\frac{K(z,w)}{h}} + \mathcal{O}(h^\infty).$$

By Taylor expansion we conclude the following result:

**Corollary 31.** *Under the assumptions of Proposition 30, we have that for  $N \geq 1$  and  $|z - w| \geq \frac{h^N}{\mathcal{O}(1)}$*

$$\min_{\lambda \in \sigma(A)} \lambda \geq \frac{h^{2N-1}}{\mathcal{O}(1)}.$$

**5.2. The matrix  $\Gamma$ .** The principal aim of this section is to prove a precise formula for the matrix  $\Gamma$ , see Proposition 33 below, and to give formulas for its determinant, permanent and trace, see Corollary 34 below.

We begin by considering a very helpful congruency transformation. In view of Proposition 19, we prove

**Lemma 32.** *Let  $\Omega \Subset \Sigma$  be as in (1.9), and let  $\Delta_\Omega(C)$ ,  $\Phi(z; h)$  and  $\Psi(z, w; h)$  be as in Proposition 19, for  $(z, w) \in \Delta_\Omega(C)$ . Let  $\Gamma$  be as in (5.1). Define the matrices*

$$\tilde{A} := \begin{pmatrix} e^{\frac{2}{h}\Psi(z,z;h)} & e^{\frac{2}{h}\Psi(z,w;h)} \\ e^{\frac{2}{h}\Psi(w,z;h)} & e^{\frac{2}{h}\Psi(w,w;h)} \end{pmatrix} \quad \text{and} \quad \Lambda := \begin{pmatrix} e^{-\frac{1}{h}\Phi(z;h)} & 0 \\ 0 & e^{-\frac{1}{h}\Phi(w;h)} \end{pmatrix},$$

$$\tilde{B} := 2h^{-1} \begin{pmatrix} \Psi'_{\bar{w}}(z, z; h) e^{\frac{2}{h}\Psi(z,z;h)} & \Psi'_{\bar{w}}(z, w; h) e^{\frac{2}{h}\Psi(z,w;h)} \\ \Psi'_{\bar{w}}(w, z; h) e^{\frac{2}{h}\Psi(w,z;h)} & \Psi'_{\bar{w}}(w, w; h) e^{\frac{2}{h}\Psi(w,w;h)} \end{pmatrix}$$

and

$$\tilde{C} := h^{-2} \begin{pmatrix} c(z, z; h) e^{\frac{2}{h}\Psi(z,z;h)} & c(z, w; h) e^{\frac{2}{h}\Psi(z,w;h)} \\ c(w, z; h) e^{\frac{2}{h}\Psi(w,z;h)} & c(w, w; h) e^{\frac{2}{h}\Psi(w,w;h)} \end{pmatrix}$$

with  $c(z, w; h) := 4\Psi'_z(z, w; h)\Psi'_{\bar{w}}(z, w; h) + 2h\Psi''_{z\bar{w}}(z, w; h)$ . Then, we have for  $|z - w| \geq h^N/\mathcal{O}(1)$  that

$$\Gamma = \Lambda(\tilde{C} - \tilde{B}^* \tilde{A}^{-1} \tilde{B})\Lambda + \mathcal{O}_{C^\infty}(h^\infty).$$

*Proof.* To abbreviate the notation, we define for  $(z, w) \in D_\Omega(C)$  the following function

$$F(z, w) := e^{-\frac{1}{h}\Phi(z;h)} e^{-\frac{1}{h}\Phi(w;h)} e^{\frac{2}{h}\Psi(z,w;h)}.$$

By Proposition 19, we see that  $F$  is bounded by 1 and that all its derivatives are bounded polynomially in  $h^{-1}$ . Furthermore, the matrices  $A, B$  and  $C$  are given by

$$\begin{aligned} A(z, w) &= A_0(z, w) + \mathcal{O}_{C^\infty}(h^\infty), \\ B(z, w) &= B_0(z, w) + \mathcal{O}_{C^\infty}(h^\infty), \\ C(z, w) &= C_0(z, w) + \mathcal{O}_{C^\infty}(h^\infty), \end{aligned}$$

where  $(z, w) \in D_\Omega(C)$  and

$$A_0(z, w) = \begin{pmatrix} F(z, z) & F(z, w) \\ F(w, z) & F(w, w) \end{pmatrix},$$

and

$$B_0(z, w) = \begin{pmatrix} (\partial_{\bar{w}} F)(z, z) & (\partial_{\bar{w}} F)(z, w) \\ (\partial_{\bar{w}} F)(w, z) & (\partial_{\bar{w}} F)(w, w) \end{pmatrix},$$

and

$$C_0(z, w) = \begin{pmatrix} (\partial_{z\bar{w}}^2 F)(z, z) & (\partial_{z\bar{w}}^2 F)(z, w) \\ (\partial_{z\bar{w}}^2 F)(w, z) & (\partial_{z\bar{w}}^2 F)(w, w) \end{pmatrix}.$$

One computes that

$$\begin{aligned} (\partial_{\bar{w}} F)(z, w) &= \frac{1}{h} [2(\partial_{\bar{w}} \Psi)(z, w; h) - (\partial_{\bar{w}} \Phi)(w; h)] e^{-\frac{1}{h}\Phi(z;h) - \frac{1}{h}\Phi(w;h)} e^{\frac{2}{h}\Psi(z,w)} \\ &\quad + \mathcal{O}_{C^\infty}(h^\infty), \end{aligned}$$

and that

$$\begin{aligned} (\partial_{z\bar{w}}^2 F)(z, w) &= \frac{1}{h^2} \left[ [2(\partial_z \Psi)(z, w; h) - (\partial_z \Phi)(z; h)] [2(\partial_{\bar{w}} \Psi)(z, w; h) - (\partial_{\bar{w}} \Phi)(w; h)] + \right. \\ &\quad \left. 2h(\partial_{z\bar{w}}^2 \Psi)(z, w; h) \right] e^{-\frac{1}{h}\Phi(z_1;h) - \frac{1}{h}\Phi(z_2;h)} e^{\frac{2}{h}\Psi(z_1, z_2)} + \mathcal{O}_{C^\infty}(h^\infty). \end{aligned}$$

Using that  $\det A_0 = \det A + \mathcal{O}(h^\infty)$  and that  $\det A \geq h^{2N-1}/\mathcal{O}(1)$  for  $|z - w| \geq h^N/\mathcal{O}(1)$  (cf. Proposition 30), we see that

$$\Gamma = C_0 - B_0^* A_0^{-1} B_0 + \mathcal{O}(h^\infty).$$

Defining,

$$\Lambda' := \begin{pmatrix} \partial_z e^{-\frac{1}{h}\Phi(z;h)} & 0 \\ 0 & \partial_w e^{-\frac{1}{h}\Phi(w;h)} \end{pmatrix}$$

we see that

$$\begin{aligned} A_0 &= \Lambda \tilde{A} \Lambda, \\ B_0 &= \Lambda(\tilde{B})\Lambda + \Lambda \tilde{A}(\Lambda') + \mathcal{O}_{C^\infty}(h^\infty), \\ C_0 &= \Lambda(\tilde{C})\Lambda + \Lambda(\tilde{B}^*)(\Lambda') + \Lambda'(\tilde{B})\Lambda + \Lambda' \tilde{A}(\Lambda') + \mathcal{O}_{C^\infty}(h^\infty). \end{aligned}$$

A direct computation then yields that

$$\Gamma = \Lambda(\tilde{C} - \tilde{B}^* \tilde{A}^{-1} \tilde{B})\Lambda + \mathcal{O}_{\mathcal{C}^\infty}((\det A)^{-1} h^\infty). \quad \square$$

**Proposition 33.** *Let  $\Omega \Subset \Sigma$  be as in (1.9), and let  $\Delta_\Omega(C)$  and  $\Psi(z, w; h)$ , for  $(z, w) \in \Delta_\Omega(C)$ , be as in Proposition 19. Let  $\Gamma$  be as in (5.1). For  $(z, w) \in D_\Omega(C)$  let  $K(z, w)$  be as in (4.2) and define*

$$\begin{aligned} a_1 &:= a_1(z, w; h) := (\partial_z \Psi)(z, z; h) - (\partial_z \Psi)(z, w; h), \\ a_2 &:= a_2(z, w; h) := -a_1(w, z; h). \end{aligned}$$

Then, for  $N > 1$  and  $\frac{1}{C} h^N \leq |z - w|$ , with  $C > 1$  large enough, we have that

$$\begin{aligned} \Gamma &= \frac{-4}{h^2 \left(1 - e^{-\frac{2}{h} K(z, w)}\right)} \begin{pmatrix} a_1 \bar{a}_1 e^{-\frac{2}{h} K(z, w)} & a_1 \bar{a}_2 e^{\frac{1}{h} (2i \operatorname{Im} \Psi(z, w) - K(z, w))} \\ a_2 \bar{a}_1 e^{\frac{1}{h} (-2i \operatorname{Im} \Psi(z, w) - K(z, w))} & a_2 \bar{a}_2 e^{-\frac{2}{h} K(z, w)} \end{pmatrix} \\ &\quad + \frac{2}{h} \begin{pmatrix} \Psi''_{z\bar{w}}(z, z; h) & \Psi''_{z\bar{w}}(z, w; h) e^{\frac{1}{h} (2i \operatorname{Im} \Psi(z, w) - K(z, w))} \\ \Psi''_{z\bar{w}}(w, z; h) e^{\frac{1}{h} (-2i \operatorname{Im} \Psi(z, w) - K(z, w))} & \Psi''_{z\bar{w}}(w, w; h) \end{pmatrix} \\ &\quad + \mathcal{O}(h^\infty). \end{aligned}$$

Before we give the proof of this result, we state formulae for the trace, the determinant and the permanent of  $\Gamma$ .

**Corollary 34.** *Under the assumptions of Proposition 33, we have that*

$$\begin{aligned} \operatorname{tr} \Gamma &= \frac{2}{h \left(e^{\frac{2}{h} K(z, w)} - 1\right)} \left[ \left( \Psi''_{z\bar{w}}(z, z; h) + \Psi''_{z\bar{w}}(w, w; h) + \mathcal{O}(h^\infty) \right) \left( e^{\frac{2}{h} K(z, w)} - 1 \right) \right. \\ &\quad \left. - 2h^{-1} (|a_1|^2 + |a_2|^2) \right], \end{aligned}$$

$$\begin{aligned} \det \Gamma &= -\frac{16}{h^4 \left(1 - e^{-\frac{2}{h} K(z, w)}\right)} e^{-\frac{2}{h} K(z, w)} \left[ |a_1 a_2|^2 + \frac{h}{2} (|a_1|^2 (\partial_{z\bar{w}}^2 \Psi)(w, w; h) \right. \\ &\quad \left. - 2 \operatorname{Re} \{ (\partial_{z\bar{w}}^2 \Psi)(w, z; h) a_1 \bar{a}_2 \} + |a_2|^2 (\partial_{z\bar{w}}^2 \Psi)(z, z; h) \right] \\ &\quad + \frac{4}{h^2} \left( (\partial_{z\bar{w}}^2 \Psi)(z, z; h) (\partial_{z\bar{w}}^2 \Psi)(w, w; h) - (\partial_{z\bar{w}}^2 \Psi)(z, w; h) (\partial_{z\bar{w}}^2 \Psi)(w, z; h) e^{-\frac{2}{h} K(z, w)} \right) \\ &\quad + \mathcal{O}(h^\infty) \end{aligned}$$

and that

$$\begin{aligned} \text{perm } \Gamma &= \frac{16}{h^4 \left(1 - e^{-\frac{2}{h}K(z,w)}\right)^2} e^{-\frac{2}{h}K(z,w)} |a_1 a_2|^2 \left(1 + e^{-\frac{2}{h}K(z,w)}\right) \\ &\quad - \frac{8}{h^3 \left(1 - e^{-\frac{2}{h}K(z,w)}\right)} e^{-\frac{2}{h}K(z,w)} (|a_1|^2 (\partial_{z\bar{w}}^2 \Psi)(w, w; h) \\ &\quad + 2\text{Re} \{ (\partial_{z\bar{w}}^2 \Psi)(w, z; h) a_1 \bar{a}_2 \} + |a_2|^2 (\partial_{z\bar{w}}^2 \Psi)(z, z; h)) \\ &\quad + \frac{4}{h^2} \left( (\partial_{z\bar{w}}^2 \Psi)(z, z; h) (\partial_{z\bar{w}}^2 \Psi)(w, w; h) + (\partial_{z\bar{w}}^2 \Psi)(z, w; h) (\partial_{z\bar{w}}^2 \Psi)(w, z; h) e^{-\frac{2}{h}K(z,w)} \right) \\ &\quad + \mathcal{O}(h^\infty). \end{aligned}$$

*Proof.* The result follows from a direct computation using Proposition 33; for the definition of the permanent of a matrix see (3.15).  $\square$

of Proposition 33. In view of Lemma 32, it remains to consider the matrix

$$\tilde{\Gamma} := \tilde{C} - \tilde{B}^* \tilde{A}^{-1} \tilde{B}.$$

In the sequel we will suppress the  $h$ -dependency of the function  $\Psi$  to abbreviate our notation. Recall the definition of  $\tilde{A}$  from Lemma 32 and note that

$$\begin{aligned} \det \tilde{A} &= e^{\frac{2}{h}\Psi(z,z)} e^{\frac{2}{h}\Psi(w,w)} - e^{\frac{4}{h}\text{Re } \Psi(z,w)} \\ &= e^{\frac{2}{h}\Psi(z,z)} e^{\frac{2}{h}\Psi(w,w)} \left(1 - e^{-\frac{2}{h}K(z,w)}\right). \end{aligned} \quad (5.2)$$

For  $\frac{1}{C}h^N \leq |z - w|$ , Proposition 19 implies that  $\det \tilde{A}$  is positive. Hence, the inverse of  $\tilde{A}$  exists and is given by

$$\tilde{A}^{-1} := \frac{1}{\det \tilde{A}} \begin{pmatrix} e^{\frac{2}{h}\Psi(w,w)} & -e^{\frac{2}{h}\Psi(z,w)} \\ -e^{\frac{2}{h}\Psi(w,z)} & e^{\frac{2}{h}\Psi(z,z)} \end{pmatrix}.$$

To calculate  $\tilde{B}^*$ , we use Lemma 32 and the symmetries of the function  $\Psi(z, w)$  given in Proposition 19. Indeed, one gets that

$$\tilde{B}^* := 2h^{-1} \begin{pmatrix} \Psi'_z(z, z) e^{\frac{2}{h}\Psi(z,z)} & \Psi'_z(z, w) e^{\frac{2}{h}\Psi(z,w)} \\ \Psi'_z(w, z) e^{\frac{2}{h}\Psi(w,z)} & \Psi'_z(w, w) e^{\frac{2}{h}\Psi(w,w)} \end{pmatrix}$$

and one computes that  $M := h\tilde{B}^* \tilde{A}^{-1} h\tilde{B}$  is given by

$$M = \frac{4}{\det \tilde{A}} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

with

$$\begin{aligned} M_{11} &= \Psi'_z(z, z) \Psi'_{\bar{w}}(z, z) e^{\frac{1}{h}(4\Psi(z,z) + 2\Psi(w,w))} + [\Psi'_z(z, w) \Psi'_{\bar{w}}(w, z) \\ &\quad - \Psi'_z(z, w) \Psi'_{\bar{w}}(z, z) - \Psi'_z(z, z) \Psi'_{\bar{w}}(w, z)] e^{\frac{1}{h}(2\Psi(z,z) + 4\text{Re } \Psi(z,w))}, \\ M_{12} &= -\Psi'_z(z, w) \Psi'_{\bar{w}}(z, w) e^{\frac{1}{h}(4\Psi(z,w) + 2\Psi(w,z))} + [\Psi'_z(z, z) \Psi'_{\bar{w}}(z, w) \\ &\quad + \Psi'_z(z, w) \Psi'_{\bar{w}}(w, w) - \Psi'_z(z, z) \Psi'_{\bar{w}}(w, w)] e^{\frac{2}{h}(\Psi(z,z) + \Psi(z,w) + \Psi(w,w))}, \end{aligned}$$

and

$$M_{22} = \Psi'_z(w, w) \Psi'_{\bar{w}}(w, w) e^{\frac{1}{h}(2\Psi(z, z) + 4\Psi(w, w))} + [\Psi'_z(w, z) \Psi'_{\bar{w}}(z, w) - \Psi'_z(w, w) \Psi'_{\bar{w}}(z, w) - \Psi'_z(w, z) \Psi'_{\bar{w}}(w, w)] e^{\frac{1}{h}(2\Psi(w, w) + 4\operatorname{Re} \Psi(z, w))}.$$

Since the matrix  $M$  is clearly self-adjoint, one has that  $M_{21} = \overline{M}_{12}$ . Comparing the coefficients of  $M$  with those of  $h^2(\det \tilde{A}/4)\tilde{C}$  (cf. Lemma 32) and using the symmetries of  $\Psi$  (cf. Proposition 19), we see that

$$h^2 \tilde{\Gamma} = \frac{-4}{\det \tilde{A}} \begin{pmatrix} a_1 \bar{a}_1 e^{\frac{1}{h}(2\Psi(z, z) + 4\operatorname{Re} \Psi(z, w))} & a_1 \bar{a}_2 e^{\frac{2}{h}(\Psi(z, z) + \Psi(z, w) + \Psi(w, w))} \\ a_2 \bar{a}_1 e^{\frac{2}{h}(\Psi(z, z) + \Psi(w, z) + \Psi(w, w))} & a_2 \bar{a}_2 e^{\frac{1}{h}(2\Psi(w, w) + 4\operatorname{Re} \Psi(z, w))} \end{pmatrix} + 2h \begin{pmatrix} \Psi''_{z\bar{w}}(z, z; h) e^{\frac{2}{h}\Psi(z, z)} & \Psi''_{z\bar{w}}(z, w; h) e^{\frac{2}{h}\Psi(z, w)} \\ \Psi''_{z\bar{w}}(w, z; h) e^{\frac{2}{h}\Psi(w, z)} & \Psi''_{z\bar{w}}(w, w; h) e^{\frac{2}{h}\Psi(w, w)} \end{pmatrix} \quad (5.3)$$

with  $a_i$  as in the hypothesis of Proposition 33. Recall from (4.2) that the function  $K(z, w)$  is defined by

$$-K(z, w) = 2\operatorname{Re} \Psi(z, w) - \Phi(z) - \Phi(w)$$

where  $\Phi(z) = \Psi(z, z)$ . Using (5.2), we find that the first matrix in (5.3) is equal to

$$\frac{-4}{1 - e^{-\frac{2}{h}K(z, w)}} \begin{pmatrix} a_1 \bar{a}_1 e^{\frac{1}{h}(2\Psi(z, z) - 2K(z, w))} & a_1 \bar{a}_2 e^{\frac{2}{h}\Psi(z, w)} \\ a_2 \bar{a}_1 e^{\frac{2}{h}\Psi(w, z)} & a_2 \bar{a}_2 e^{\frac{1}{h}(2\Psi(w, w) - 2K(z, w))} \end{pmatrix}.$$

It follows by Lemma 32 that

$$\Gamma = \Lambda \tilde{\Gamma} \Lambda^* + \mathcal{O}_{C^\infty}(h^\infty).$$

In the last equality we used that  $\det A$  is bounded from below by a power of  $h$ ; see Lemma 32. Carrying out the matrix multiplication  $\Lambda \tilde{\Gamma} \Lambda^*$  implies the statement of the proposition.  $\square$

**5.3. The determinant of  $G$ .** We show that the matrix  $G(z, w)$  (cf. (3.12)) is invertible if  $z$  and  $w$  are outside a neighborhood of size of order  $h^{3/5}$  of the diagonal  $\{z = w\}$ . More precisely, we prove the following result:

**Proposition 35.** *Let  $\Omega \Subset \Sigma$  be as in (1.9) and let  $z, w \in \Omega$ . Then,*

$$\det G(z, w) > 0 \quad \text{for } h^{\frac{3}{5}} \ll |z - w| \ll 1.$$

*Proof.* The Shur complement formula yields that the determinant of the Gramian matrix  $G$  is given by  $\det G = \det A \det \Gamma$ . Hence, using Proposition 30 and Corollary 34, we see that

$$\begin{aligned} \det G = & -\frac{16(1 + \mathcal{O}(h^\infty))}{h^4} e^{-\frac{2}{h}K(z, w)} \left[ |a_1 a_2|^2 + \frac{h}{2} (|a_1|^2 (\partial_{z\bar{w}}^2 \Psi)(w, w; h) \right. \\ & \left. - 2\operatorname{Re} \{ (\partial_{z\bar{w}}^2 \Psi)(w, z; h) a_1 \bar{a}_2 \} + |a_2|^2 (\partial_{z\bar{w}}^2 \Psi)(z, z; h) ) \right] \\ & + \frac{4}{h^2} \left( (\partial_{z\bar{w}}^2 \Psi)(z, z; h) (\partial_{z\bar{w}}^2 \Psi)(w, w; h) - (\partial_{z\bar{w}}^2 \Psi)(z, w; h) (\partial_{z\bar{w}}^2 \Psi)(w, z; h) e^{-\frac{2}{h}K(z, w)} \right) \\ & \cdot \left( 1 - e^{-\frac{2}{h}K(z, w)} + \mathcal{O}(h^\infty) \right) + \mathcal{O}(h^\infty). \end{aligned} \quad (5.4)$$

Next, we consider the Taylor expansion of the terms  $a_1$  and  $a_2$  up to first order. Similarly as in Proposition 19, we develop around the point  $(\frac{z+w}{2}, \frac{z+w}{2})$  and get that

$$\begin{aligned} a_1 &= (\partial_z \Psi)(z, z) - (\partial_z \Psi)(z, w) \\ &= (\partial_{z\bar{w}}^2 \Psi) \left( \frac{z+w}{2}, \frac{z+w}{2} \right) (z-w) + \mathcal{O}(|z-w|^2 + h^\infty) \end{aligned} \quad (5.5)$$

and

$$\begin{aligned} a_2 &= (\partial_z \Psi)(w, z) - (\partial_z \Psi)(w, w) \\ &= (\partial_{z\bar{w}}^2 \Psi) \left( \frac{z+w}{2}, \frac{z+w}{2} \right) (z-w) + \mathcal{O}(|z-w|^2 + h^\infty). \end{aligned} \quad (5.6)$$

Moreover, one has that for  $\zeta, \omega \in \{z, w\}$

$$(\partial_{z\bar{w}}^2 \Psi)(\zeta, \omega) = (\partial_{z\bar{w}}^2 \Psi) \left( \frac{z+w}{2}, \frac{z+w}{2} \right) + \mathcal{O}(|z-w| + h^\infty). \quad (5.7)$$

Since we suppose that  $|z-w| \gg h^{3/5}$ , the above error term is equal to  $\mathcal{O}(|z-w|)$ . Since  $\partial_{z\bar{w}}^2 \Psi$  is evaluated at a point on the diagonal, it follows from Proposition 19, that

$$\begin{aligned} (\partial_{z\bar{w}}^2 \Psi) \left( \frac{z+w}{2}, \frac{z+w}{2} \right) &= (\partial_{z\bar{z}}^2 \Phi) \left( \frac{z+w}{2}, \frac{z+w}{2} \right) \\ &= \frac{1}{4} \sigma \left( \frac{z+w}{2} \right) + \mathcal{O}(h) =: \frac{1}{4} \sigma_h(z, w). \end{aligned} \quad (5.8)$$

Plugging the above Taylor expansion into (5.4), one gets that  $\det G$  is equal to

$$\begin{aligned} &\frac{\sigma_h(z, w)^2}{4h^2} \left\{ \left[ 1 + \mathcal{O}(|z-w|) - (1 + \mathcal{O}(|z-w|)) e^{-\frac{2}{h} K(z, w)} \right] \left( 1 - e^{-\frac{2}{h} K(z, w)} + \mathcal{O}(h^\infty) \right) \right. \\ &\quad \left. - 4e^{-\frac{2}{h} K(z, w)} \left( \left( \frac{\sigma_h(z, w)|z-w|^2}{4h} \right)^2 (1 + \mathcal{O}(|z-w|)) + \frac{\sigma_h(z, w)|z-w|^2}{4h} \mathcal{O}(|z-w|) \right) \right\} \\ &\quad + \mathcal{O}(h^\infty) \\ &= \frac{\sigma_h(z, w)^2}{4h^2} \left\{ \left( 1 - e^{-\frac{2}{h} K(z, w)} \right)^2 + \mathcal{O}(|z-w|) \left( 1 - e^{-\frac{2}{h} K(z, w)} \right) + \mathcal{O}(h^\infty) \right. \\ &\quad \left. - 4e^{-\frac{2}{h} K(z, w)} \left[ \left( \frac{\sigma_h(z, w)|z-w|^2}{4h} \right)^2 + \mathcal{O} \left( \frac{|z-w|^5}{h^2} \right) + \mathcal{O} \left( \frac{|z-w|^3}{h} \right) \right] \right\}. \end{aligned}$$

Recall from (4.2) that  $K(z, w) \asymp |z-w|^2$ , wherefore we see that  $\det G$  is positive for  $|z-w| \gg \sqrt{h}$ . Next, we suppose that  $|z-w| \asymp \sqrt{h}$ . Hence, one gets that

$$\begin{aligned} \det G &= \frac{\sigma_h(z, w)^2 e^{-\frac{2}{h} K(z, w)}}{h^2} \left\{ \sinh^2 \frac{K(z, w)}{h} + \mathcal{O}(|z-w|) \left( e^{\frac{2}{h} K(z, w)} - 1 \right) + \mathcal{O}(h^\infty) \right. \\ &\quad \left. - \left[ \left( \frac{\sigma_h(z, w)|z-w|^2}{4h} \right)^2 + \mathcal{O} \left( \frac{|z-w|^5}{h^2} \right) + \mathcal{O} \left( \frac{|z-w|^3}{h} \right) \right] \right\}. \end{aligned} \quad (5.9)$$



Using the Taylor expansion of the  $\sinh x$  and (4.2), one gets that

$$\begin{aligned} & \sinh^2 \frac{K(z, w)}{h} - \left( \frac{\sigma_h(z, w)|z - w|^2}{4h} \right)^2 \\ & \geq \left( \frac{1}{3} \frac{\sigma_h(z, w)|z - w|^2}{4h} \right)^4 (1 + \mathcal{O}(|z - w|)) + \mathcal{O}\left( \frac{\sigma_h(z, w)|z - w|^5}{h^2} \right). \end{aligned} \quad (5.10)$$

Note that the principal term on the right hand side of the inequality dominates the error terms. The same holds true for the other error terms in (5.9).

Next, let us suppose that  $h^{3/5} \ll |z - w| \ll \sqrt{h}$ . Since

$$\mathcal{O}(|z - w|) \left( e^{\frac{2}{h}K(z, w)} - 1 \right) = \mathcal{O}\left( \frac{|z - w|^3}{h} \right),$$

it follows by (5.9) and (5.10) that  $\det G$  is positive for  $|z - w| \gg h^{3/5}$ .  $\square$

**5.4. The permanent of  $\Gamma$ .** The permanent of the matrix  $\Gamma$  (cf. (5.1)) is vital to the 2-point density of eigenvalues and therefore, we shall give a more detailed description of it than the one given in Corollary 34.

We begin by proving the following bound on the trace of  $\Gamma$ :

**Proposition 36.** *Under the assumptions of Proposition 33, we have that for  $|z - w| \gg h$*

$$0 < \text{tr } \Gamma \leq \mathcal{O}(h^{-1}).$$

*Proof.* Using (5.5), (5.6) and (5.7), one gets that

$$\begin{aligned} \text{tr } \Gamma &= \frac{\sigma_h(z, w)}{2h \left( e^{\frac{2}{h}K(z, w)} - 1 \right)} \left[ \left( e^{\frac{2}{h}K(z, w)} - 1 \right) (1 + \mathcal{O}(|z - w|)) \right. \\ &\quad \left. - \frac{\sigma_h(z, w)|z - w|^2}{2h} (1 + \mathcal{O}(|z - w|)) \right]. \end{aligned} \quad (5.11)$$

Since

$$e^{\frac{2}{h}K(z, w)} - 1 \geq \frac{\sigma_h(z, w)|z - w|^2}{2h} (1 + \mathcal{O}(|z - w|)) + \frac{\sigma_h(z, w)|z - w|^4}{8h^2} (1 + \mathcal{O}(|z - w|)),$$

it follows that for  $|z - w| \gg h$  the trace of  $\Gamma$  is positive. Furthermore, the above inequality applied to (5.11), implies the upper bound stated in the Proposition.  $\square$

**Proposition 37.** *Let  $\sigma_h(z, w)$  be as in Theorem 5 and let  $K(z, w)$  be as in (4.2). Under the assumptions of Proposition 33, we have that for  $N > 1$  and  $\frac{1}{C}h^N \leq |z - w|$ ,*

$\text{perm } \Gamma(z, w; h)$

$$\begin{aligned} &= \frac{1}{4h^2} \left[ \sigma_h(z, z)\sigma_h(w, w) + \sigma_h(z, w)^2 (1 + \mathcal{O}(|z - w|)) e^{-\frac{2K(z, w)}{h}} + \mathcal{O}(h^\infty) \right. \\ &\quad \left. + \frac{\sigma_h(z, w)^2 (1 + \mathcal{O}(|z - w|))}{e^{\frac{K(z, w)}{h}} \sinh \frac{K(z, w)}{h}} \left( \left( \frac{\sigma_h(z, w)|z - w|^2}{4h} \right)^2 2 \coth \frac{K(z, w)}{h} - \frac{\sigma_h(z, w)|z - w|^2}{h} \right) \right]. \end{aligned}$$

*Proof.* Applying (5.5), (5.6) and (5.7) to the formula for  $\text{perm } \Gamma$  given in Proposition 36 and using the notation introduced in (5.8), one gets that

$$\begin{aligned} \text{perm } \Gamma &= \frac{8 \coth \frac{K}{h}}{h^4 \sinh \frac{K}{h}} e^{-\frac{1}{h}K(z,w)} |4^{-2} \sigma_h(z, w)^2 (z - w)^2 (1 + \mathcal{O}(|z - w|))^2 \\ &\quad - \frac{e^{-\frac{1}{h}K(z,w)}}{4h^3 \sinh \frac{K}{h}} \sigma_h(z, w)^3 |z - w|^2 (1 + \mathcal{O}(|z - w|)) \\ &\quad + \frac{1}{4h^2} \left( \sigma_h(z, z) \sigma_h(w, w) + \sigma_h(z, w; h)^2 (1 + \mathcal{O}(|z - w|) e^{-\frac{2}{h}K(z,w)}) \right) \\ &\quad + \mathcal{O}(h^\infty). \end{aligned}$$

Thus, one computes that

$$\begin{aligned} \text{perm } \Gamma &= \frac{\sigma_h(z, w)^2 (1 + \mathcal{O}(|z - w|))}{4h^2 e^{\frac{1}{h}K(z,w)} \sinh \frac{K}{h}} \left[ \left( \frac{\sigma_h(z, w) |z - w|^2}{4h} \right)^2 2 \coth \frac{K}{h} - \frac{\sigma_h(z, w) |z - w|^2}{h} \right] \\ &\quad + \frac{1}{4h^2} \left( \sigma_h(z, z) \sigma_h(w, w) + \sigma_h(z, w; h)^2 (1 + \mathcal{O}(|z - w|) e^{-\frac{2}{h}K(z,w)}) \right) \\ &\quad + \mathcal{O}(h^\infty) \end{aligned}$$

and we conclude the statement of the proposition.  $\square$

## 6. PROOF OF PROPOSITION 17

The first ingredient of the proof of Proposition 17, is the following global version of the implicit function theorem.

**Lemma 38.** *Let  $0 < R_0 < R$ , let  $n, m \in \mathbb{N}$ , with  $n > m$ , and let  $B(0, R) \subset \mathbb{C}^n = \mathbb{C}_z^{n-m} \times \mathbb{C}_w^m$  denote the complex open ball of radius  $R > 0$  centered at 0. For  $z \in B_{\mathbb{C}^{n-m}}(0, R_0)$ , define  $R(z) := (R^2 - \|z\|_{\mathbb{C}^{n-m}}^2)^{1/2}$ . We consider a holomorphic function*

$$F : B(0, R) \longrightarrow \mathbb{C}^m$$

such that

- for all  $(z, w) \in B(0, R)$  the Jacobian of  $F$  with respect to  $w$  is given by

$$\frac{\partial F(z, w)}{\partial w} = A + G(z, w),$$

where  $G : B(0, R) \longrightarrow \mathbb{C}^{m \times m}$  is a matrix-valued holomorphic function and

- $A \in \text{GL}_m(\mathbb{C})$  such that

$$\|A^{-1}\| \cdot \|G(z, w)\| \leq \theta < 1$$

for all  $(z, w) \in B(0, R)$ .

Then, for all  $z \in B_{\mathbb{C}^{n-m}}(0, R_0)$  and for all  $y \in B_{\mathbb{C}^m}(F(z, 0), \frac{1-\theta}{\|A^{-1}\|}r)$ , with  $0 < r < R(z)$ , the equation

$$F(z, w) = y \tag{6.1}$$

has exactly one solution  $w(z, y) \in B_{\mathbb{C}^m}(0, R(z))$ , it satisfies  $w(z, y) \in B_{\mathbb{C}^m}(0, r)$  and it depends holomorphically on  $z$  and on  $y$ .

**Remark 39.** Observe that the choice of  $R_0 < R$  yields a uniform lower bound on  $R(z)$  and so we can choose the radius of the ball  $B_{\mathbb{C}^m}(F(z, 0), \frac{1-\theta}{\|A^{-1}\|}r)$  uniformly in  $z$ . This will become important in the proof of Proposition 17.

*Proof.* Let  $z \in B_{\mathbb{C}^{n-m}}(0, R_0)$  and set

$$B_{\mathbb{C}^m}(0, R(z)) \ni w \longmapsto \tilde{F}(w) := F(z, w).$$

We begin by observing that  $d\tilde{F}(w)$  is invertible for all  $w \in B_{\mathbb{C}^m}(0, R(z))$  and the norm of the inverse is bounded (uniformly in  $z$ ). Indeed, for one has that

$$\left\| \left( d\tilde{F}(w) \right)^{-1} \right\| \leq \|A^{-1}\| \cdot \|(1 + A^{-1}G(z, w))^{-1}\| \leq \frac{\|A^{-1}\|}{1-\theta}.$$

*Claim #1:*  $\tilde{F}$  is injective.

Let  $w_0, w_1 \in B_{\mathbb{C}^m}(0, R(z))$  and define  $y_i := \tilde{F}(w_i)$ . Hence, with  $w_t := (1-t)w_0 + tw_1$ , we have that

$$\frac{d}{dt}\tilde{F}(w_t) = d\tilde{F}(w_t) \cdot (w_1 - w_0) = (A + G(z, w_t)) \cdot (w_1 - w_0).$$

Thus,

$$y_1 - y_0 = (A + H(z, w_1, w_0)) \cdot (w_1 - w_0), \quad H(z, w_1, w_0) = \int_0^1 G(z, w_t) dt,$$

where  $\|H(z, w_1, w_0)\| \leq \sup_{B(0, R)} \|G(z, w)\|$ . Therefore,  $\|A^{-1}\| \cdot \|H(z, w_1, w_0)\| \leq \theta < 1$ , and we see that  $(A + H(z, w_1, w_0))$  is invertible and the norm of its inverse is  $\leq \frac{\|A^{-1}\|}{1-\theta}$  (uniformly in  $z$ ). Hence,

$$\|w_1 - w_0\| \leq \frac{\|A^{-1}\|}{1-\theta} \|y_1 - y_0\|, \tag{6.2}$$

and we conclude that  $\tilde{F}$  is injective. In particular, we have proven the uniqueness of the solution to the equation (6.1).

*Claim #2:* Let  $0 < r < R(z)$ . Then, for all  $y \in B_{\mathbb{C}^m}(\tilde{F}(0), \frac{1-\theta}{\|A^{-1}\|}r)$  there exists a  $w \in B_{\mathbb{C}^m}(0, r)$  such that

$$\tilde{F}(w) = y.$$

For  $y = \tilde{F}(0)$ , we take  $w = 0$ . Using the fact that  $d\tilde{F}$  is invertible everywhere, the implicit function theorem implies that for all  $y \in B(\tilde{F}(0), \rho)$  there exists a solution  $w \in B_{\mathbb{C}^m}(0, r)$ , if  $\rho > 0$  is small enough (cf. (6.2)). Let  $y \in B_{\mathbb{C}^m}(\tilde{F}(0), \frac{1-\theta}{\|A^{-1}\|}r)$ , and define  $y_t := (1-t)\tilde{F}(0) + ty$ . Let  $t_0 \in [0, 1]$  be the supremum of  $\tilde{t} \in [0, 1]$  such that there exists a solution to  $\tilde{F}(w_t) = y_t$  for all  $0 \leq t \leq \tilde{t}$ .

We have already proven that  $t_0 > 0$ . As  $t \nearrow t_0$  we have that  $w_t \in B_{\mathbb{C}^m}(0, r)$ . Since  $B_{\mathbb{C}^m}(0, r)$  is relatively compact in  $B_{\mathbb{C}^m}(0, R(z))$ , there exists a sequence  $t_j \nearrow t_0$  such that  $w_{t_j} \rightarrow \tilde{w}$  with  $\tilde{w} \in \overline{B_{\mathbb{C}^m}(0, r)}$ . Thus,

$$\tilde{F}(\tilde{w}) = y_{t_0},$$

and we see by (6.2) that  $\tilde{w} \in B_{\mathbb{C}^m}(0, r)$ .

If  $t_0 < 1$ , we get by the implicit function theorem, that for all  $y \in B(y_{t_0}, \delta)$ , with  $\delta > 0$  small enough, there exists a solution  $w \in B_{\mathbb{C}^m}(0, r)$ . Therefore, we can solve  $\tilde{F}(w_t) = y_t$  for all  $0 < t < t_0 + \delta$ , which is a contradiction. Hence,  $t_0 = 1$ , which concludes the proof of the existence of a solution.

Finally, note that for all  $(z, w) \in B(0, R)$  the Jacobian  $\partial F(z, w)/\partial w$  is invertible and the norm of its inverse is uniformly bounded, indeed

$$\left\| \left( \frac{\partial F(z, w)}{\partial w} \right)^{-1} \right\| \leq \|A^{-1}\| \cdot \|(1 + A^{-1}G(z, w))^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \theta}.$$

In particular, we have that the determinant of the Jacobian is never equal to 0, and we conclude by the holomorphic implicit function theorem that the solution  $w(z, y)$  to the equation (6.2) depends holomorphically on  $z$  and  $y$ .  $\square$

of Proposition 17. In view of (3.10), it remains to study the integral

$$I(z_1, z_2, h) = \lim_{\varepsilon \rightarrow 0^+} \pi^{-N} \int_{B(0, R)} H_\varepsilon^\delta(z_1, z_2, \alpha; h) e^{-\alpha \bar{\alpha}} L(d\alpha). \quad (6.3)$$

with

$$H_\varepsilon^\delta(z_1, z_2, \alpha; h) := \prod_{k=1}^2 \varepsilon^{-2} \chi \left( \frac{E_{-+}^\delta(z_k, \alpha)}{\varepsilon} \right) |\partial_{z_k} E_{-+}^\delta(z_k, \alpha)|^2$$

for  $1/C \geq |z_1 - z_2| \gg h^{3/5}$ . We begin by performing a change of variables in the  $\alpha$ -space.

**Change of variables:** For  $X(z) \in \mathbb{C}^N$  as in Definition 12, define the matrix

$${}^tV := (X(z_1), X(z_2), \partial_{z_1} X(z_1), \partial_{z_2} X(z_2)) \in \mathbb{C}^{N \times 4}$$

and note that the Gramian matrix  $G$  (cf. (3.12)) satisfies

$$G = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = V \cdot V^*.$$

Moreover,  $G$  is invertible by virtue of Proposition 35, since  $|z_1 - z_2| \gg h^{3/5}$ . Next, we define the matrix  $U \in \mathbb{C}^{4 \times 4}$  by

$$U := \begin{pmatrix} 1 & 0 \\ B^* A^{-1} & 1 \end{pmatrix}.$$

$U$  is invertible and thus satisfies that  $(U^{-1})^* = (U^*)^{-1}$ . Define the matrix

$$\tilde{G} := \begin{pmatrix} A & 0 \\ 0 & \Gamma \end{pmatrix} \in \mathbb{C}^{4 \times 4},$$

and notice that

$$U \begin{pmatrix} A & 0 \\ 0 & \Gamma \end{pmatrix} U^* = \begin{pmatrix} 1 & 0 \\ B^* A^{-1} & 1 \end{pmatrix} \tilde{G} \begin{pmatrix} 1 & A^{-1} B \\ 0 & 1 \end{pmatrix} = G.$$

We see that  $\tilde{G} = U^{-1} G (U^*)^{-1}$ . Next, we define the matrix

$$\tilde{V}^* := (U^{-1} V)^* \tilde{G}^{-\frac{1}{2}} \in \mathbb{C}^{N \times 4}. \quad (6.4)$$

$\tilde{V}^*$  is an isometry since  $\tilde{V}\tilde{V}^* = 1_{\mathbb{C}^4}$ . Thus, its columns form an orthonormal family in  $\mathbb{C}^N$ . It follows from (6.4) that the kernel of  $V$  and of  $\tilde{V}$  are equal, i.e.  $\mathcal{N}(V) = \mathcal{N}(\tilde{V})$ . The same holds true for the range of  $\tilde{V}$  and of  $V$ , i.e.  $\mathcal{R}(V) = \mathcal{R}(\tilde{V})$ .

Next, we choose an orthonormal basis,  $e_1, \dots, e_N \in \mathbb{C}^N$ , of the space of random variables  $\alpha$  such that  $\tilde{V}_1^*, \dots, \tilde{V}_4^*$ , the column vectors of the matrix  $\tilde{V}^*$ , are among them. In particular, let  $e_i = \tilde{V}_i^*$  for  $i = 1, \dots, 4$ , and let  $e_5, \dots, e_N$  be in the orthogonal complement of the space spanned by  $e_1, \dots, e_4$ . Hence, we write for  $\alpha \in \mathbb{C}^N$

$$\alpha = \sum_{i=1}^N \tilde{\alpha}_i e_i,$$

where  $\tilde{\alpha} = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_N) \in \mathbb{C}^N$ . Moreover, note that

$$\alpha^* \cdot \alpha = \tilde{\alpha}^* \cdot \tilde{\alpha}. \quad (6.5)$$

**Remark 40.** Recall from Proposition 35 that we can only guarantee the invertibility of  $G$  for  $h^{\frac{3}{5}} \ll |z - w| \ll 1$ . This makes the assumption in Proposition 17 (and Theorem 5) that the support of the test function  $\varphi$  avoids  $D(\Omega, c)$ , see (2.2), necessary. This might be avoided by choosing another set of basis vectors.

Next, we apply this change of variables to the vector  $F$  given in (3.11) and we get

$$\begin{aligned} F(z, \alpha(\tilde{\alpha}); \delta, h) &= \begin{pmatrix} E_{-+}(z_1) \\ E_{-+}(z_2) \\ (\partial_z E_{-+})(z_1) \\ (\partial_z E_{-+})(z_2) \end{pmatrix} - \delta \begin{pmatrix} {}^t X(z_1) \\ {}^t X(z_2) \\ {}^t (\partial_z X)(z_1) \\ {}^t (\partial_z X)(z_2) \end{pmatrix} \cdot \alpha(\tilde{\alpha}) + \begin{pmatrix} T(z_1, \alpha(\tilde{\alpha})) \\ T(z_2, \alpha(\tilde{\alpha})) \\ (\partial_z T)(z_1, \alpha(\tilde{\alpha})) \\ (\partial_z T)(z_2, \alpha(\tilde{\alpha})) \end{pmatrix} \\ &= \begin{pmatrix} E_{-+}(z_1) \\ E_{-+}(z_2) \\ (\partial_z E_{-+})(z_1) \\ (\partial_z E_{-+})(z_2) \end{pmatrix} - \delta(V \cdot \tilde{V}) \cdot \begin{pmatrix} \tilde{\alpha}_1 \\ \vdots \\ \tilde{\alpha}_4 \end{pmatrix} + \begin{pmatrix} T(z_1, \alpha(\tilde{\alpha})) \\ T(z_2, \alpha(\tilde{\alpha})) \\ (\partial_z T)(z_1, \alpha(\tilde{\alpha})) \\ (\partial_z T)(z_2, \alpha(\tilde{\alpha})) \end{pmatrix}. \end{aligned}$$

Furthermore, one computes that

$$V\tilde{V} = U\tilde{G}^{\frac{1}{2}} = \begin{pmatrix} A^{\frac{1}{2}} & 0 \\ B^* A^{-\frac{1}{2}} & \Gamma^{\frac{1}{2}} \end{pmatrix}, \quad (6.6)$$

and we get that

$$F(z, \alpha(\tilde{\alpha}); \delta, h) = \begin{pmatrix} E_{-+}(z_1) \\ E_{-+}(z_2) \\ (\partial_z E_{-+})(z_1) \\ (\partial_z E_{-+})(z_2) \end{pmatrix} - \delta U \tilde{G}^{\frac{1}{2}} \cdot \begin{pmatrix} \tilde{\alpha}_1 \\ \vdots \\ \tilde{\alpha}_4 \end{pmatrix} + \begin{pmatrix} T(z_1, \alpha(\tilde{\alpha})) \\ T(z_2, \alpha(\tilde{\alpha})) \\ (\partial_z T)(z_1, \alpha(\tilde{\alpha})) \\ (\partial_z T)(z_2, \alpha(\tilde{\alpha})) \end{pmatrix}.$$

Next, to simplify our notation, we call the  $\tilde{\alpha}$  variables again  $\alpha$ . Also, to abbreviate our notation, define

$$\mu(z, w; h) := \begin{pmatrix} E_{-+}(z_1) \\ E_{-+}(z_2) \end{pmatrix} \text{ and } \tau(z, \alpha; h, \delta) := \begin{pmatrix} T(z_1, \alpha) \\ T(z_2, \alpha) \end{pmatrix}.$$

and

$$\partial_z \mu(z, w; h) := \begin{pmatrix} (\partial_z E_{-+})(z_1) \\ (\partial_z E_{-+})(z_2) \end{pmatrix} \text{ and } \partial_z \tau(z, \alpha; h, \delta) := \begin{pmatrix} (\partial_z T)(z_1, \alpha) \\ (\partial_z T)(z_2, \alpha) \end{pmatrix}.$$

**Remark 41.** Recall that  $T$  (cf. (3.9)) depends on  $h$  and on  $\delta$ , though not explicit in the above notation.

When we write  $\partial_z \mu$  and  $\partial_z \tau$  the derivatives are to be understood component wise, each of which only depends either on  $z_1$  or  $z_2$ .

Hence,

$$F^\delta(z, \alpha) := F(z, \alpha; \delta, h) = \begin{pmatrix} \mu(z, h, \delta) \\ \partial_z \mu(z, h, \delta) \end{pmatrix} - \delta U \tilde{G}^{\frac{1}{2}} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_4 \end{pmatrix} + \begin{pmatrix} \tau(z, \alpha, h, \delta) \\ \partial_z \tau(z, \alpha, h, \delta) \end{pmatrix}. \quad (6.7)$$

As noted in Remark 14,  $\mu$  and  $\tau$  are smooth in  $z$ , and  $\tau$  is holomorphic in  $\alpha$ . Moreover,  $\tau$  satisfies the estimates

$$\tau_i = \mathcal{O}(h^{-5/2} \delta^2), \quad i = 1, 2 \text{ and } \partial_{z_i} \tau_i = \mathcal{O}(h^{-7/2} \delta^2), \quad i = 1, 2; \quad (6.8)$$

and  $\mu$  satisfies the estimates

$$\mu_i = \mathcal{O}(h^{1/2} e^{-\frac{S}{h}}), \quad \partial_{z_i} \mu_i = \mathcal{O}(h^{-1/2} e^{-\frac{S}{h}}), \quad i = 1, 2 \quad (6.9)$$

with  $S$  as in (1.10). Finally, we perform the above described change of variables in the integral (6.3), and, using the fact that we chose an orthonormal basis of the  $\alpha$ -space, we get that

$$H_\varepsilon^\delta(z_1, z_2, \alpha; h) = \prod_{k=1}^2 \varepsilon^{-2} \chi \left( \frac{F_k^\delta(z_k, \alpha)}{\varepsilon} \right) |F_{k+2}^\delta(z_k, \alpha)|^2.$$

Next, let  $\alpha = (\alpha_1, \alpha_2, \alpha') = (\tilde{\alpha}, \alpha')$  and split the ball  $B(0, R)$ ,  $R = Ch^{-1}$ , into two pieces: pick  $C_0 > 0$  such that  $0 < C_1 < C_0 < C < 2C_0$ , and define  $R_0 = C_0 h^{-1}$ . Then, we perform the splitting:  $I(z, h) = I_1(z, h) + I_2(z, h)$  with

$$I_1(z, h) := \lim_{\varepsilon \rightarrow 0^+} \pi^{-N} \int_{\substack{B(0, R) \\ \|\alpha'\|_{\mathbb{C}^{N-2}} \leq R_0}} H_\varepsilon^\delta(z_1, z_2, \alpha; h) e^{-\alpha^* \alpha} L(d\alpha). \quad (6.10)$$

and

$$I_2(z, h) := \lim_{\varepsilon \rightarrow 0^+} \pi^{-N} \int_{\substack{B(0, R) \\ R_0 < \|\alpha'\|_{\mathbb{C}^{N-2}} < R}} H_\varepsilon^\delta(z_1, z_2, \alpha; h) e^{-\alpha^* \alpha} L(d\alpha). \quad (6.11)$$

**The integral  $I_1$**  First, we perform a new change of variables in the  $\alpha$ -space. Let  $\beta_1, \dots, \beta_N \in \mathbb{C}$  such that

$$\beta_1 = F_1^\delta(z_1, \alpha), \quad \beta_2 = F_2^\delta(z_2, \alpha) \text{ and } \beta_i = \alpha_i, \text{ for } i = 3, \dots, N.$$

We use the following notation:  $\beta = (\beta_1, \beta_2, \beta') = (\tilde{\beta}, \alpha')$ . It is sufficient to check that we can express  $\tilde{\alpha} = (\alpha_1, \alpha_2)$  as a function of  $(\tilde{\beta}, \alpha')$ . Therefore, we apply Lemma 38 to the function

$$\mathcal{F}^\delta(z, \alpha) = \begin{pmatrix} F_1^\delta(z_1, \alpha) \\ F_2^\delta(z_2, \alpha) \end{pmatrix}.$$

where  $\alpha$  plays the role of  $(z, w)$  in the Lemma. In particular,  $\tilde{\alpha}$  plays the role of  $w$ . Let us check that the assumptions of Lemma 38 are satisfied:  $\mathcal{F}^\delta(z, \alpha)$  is by definition holomorphic in  $\alpha$ . Using (6.7) and (6.6), we see that its Jacobian, with respect to the variables  $\tilde{\alpha}$ , is given by

$$\frac{\partial \mathcal{F}(z, \alpha)}{\partial \tilde{\alpha}} = \frac{\partial \tau}{\partial \tilde{\alpha}} - \delta A^{\frac{1}{2}} \quad (6.12)$$

The Cauchy inequalities and (6.8) imply that

$$\frac{\partial \tau_i}{\partial \tilde{\alpha}_j} = \mathcal{O}\left(\delta^2 h^{-\frac{3}{2}}\right), \quad i, j = 1, 2.$$

This estimate is uniform in  $\alpha \in B(0, R)$  and  $(z_1, z_2) \in \text{supp } \varphi$ . Expansion of the determinant yields that

$$\det\left(\frac{\partial \tau}{\partial \tilde{\alpha}} - \delta A^{\frac{1}{2}}\right) = \delta^2 \left(\sqrt{\det A} + \mathcal{O}\left(\delta h^{-\frac{3}{2}}\right)\right). \quad (6.13)$$

Using that  $A$  is self-adjoint, we see by Corollary 31 that for  $(z_1, z_2) \in \text{supp } \varphi$

$$\|A^{-\frac{1}{2}}\| \leq \frac{1}{\min_{\lambda \in \sigma(A)} \sqrt{\lambda}} \leq \mathcal{O}\left(h^{-\frac{1}{10}}\right). \quad (6.14)$$

By the hypothesis (1.11), we have that  $\delta \ll h^{7/2}$ . Hence, one gets that for all  $\alpha \in B(0, R)$

$$\delta^{-1} \|A^{-\frac{1}{2}}\| \cdot \|\partial_{\tilde{\alpha}} \tau\| \leq \mathcal{O}\left(\delta h^{-\frac{3}{2} - \frac{1}{10}}\right) \ll 1.$$

Hence  $\mathcal{F}^\delta(z, \alpha)$  satisfies the assumptions of Lemma 38. In the integral  $I_1$  we restricted  $\alpha'$  to the open ball  $\|\alpha'\|_{\mathbb{C}^{N-2}} < R_0$ . It follows by Lemma 38 that for all

$$\tilde{\beta} \in B_{\mathbb{C}^2}\left(\mathcal{F}^\delta(z; 0, \alpha'), r\right) \quad (6.15)$$

with

$$\begin{aligned} r &:= \left( \delta \|A^{-\frac{1}{2}}\|^{-1} \left(1 - \max_{\alpha \in B(0, R)} \delta^{-1} \|A^{-\frac{1}{2}}\| \cdot \|\partial_{\tilde{\alpha}} \tau\| \right) \right) \sqrt{R^2 - R_0^2} \\ &\geq \frac{\delta h^{\frac{1}{10} - 1}}{\mathcal{O}(1)} > 0, \end{aligned}$$

the equation  $\tilde{\beta} = \mathcal{F}^\delta(z, \tilde{\alpha}, \alpha')$  has exactly one solution  $\tilde{\alpha}(\tilde{\beta}, \alpha'; z)$  in the ball

$$B\left(0, \sqrt{R^2 - \|\alpha'\|_{\mathbb{C}^{N-2}}^2}\right).$$

Moreover, the solution satisfies  $\tilde{\alpha}(\tilde{\beta}, \alpha'; z) \in B(0, \sqrt{R^2 - R_0^2})$ , and it depends holomorphically on  $\tilde{\beta}$  and  $\alpha'$  and is smooth in  $z$ . Using (6.7), we see that the solution is implicitly

given by

$$\tilde{\alpha}(\tilde{\beta}, \alpha') = -\delta^{-1} A^{-\frac{1}{2}} \left( \tilde{\beta} - \nu(z, \tilde{\alpha}(\tilde{\beta}, \alpha'), \alpha', h, \delta) \right). \quad (6.16)$$

with

$$\nu := (\nu_1, \nu_2)^t := \mu(z, h) + \tau(z, \tilde{\alpha}(\tilde{\beta}, \alpha'), \alpha', h, \delta)$$

where  $\tau$  satisfies the estimate (6.8). Since the support of  $\chi$  is compact (cf. Section 3.4), we can restrict our attention to  $\tilde{\beta}$  and  $\mathcal{F}^\delta(z; 0, \alpha')$  in a small poly-disc of radius  $K\varepsilon > 0$  centered at 0, with  $K > 0$  large enough such that  $\text{supp } \chi \subset D(0, K)$ . By choosing  $\varepsilon < \delta h/C$ ,  $C > 0$  large enough, we see that  $\tilde{\beta}, \mathcal{F}^\delta(z; 0, \alpha') \in D(0, K\varepsilon) \times D(0, K\varepsilon)$  implies (6.15).

From (6.6), (6.7) and (6.16), it follows that

$$\begin{pmatrix} F_3^\delta(z, \tilde{\alpha}(\tilde{\beta}, \alpha'), \alpha') \\ F_4^\delta(z, \tilde{\alpha}(\tilde{\beta}, \alpha'), \alpha') \end{pmatrix} = \partial_z \nu + B^* A^{-1} (\tilde{\beta} - \nu) - \delta \Gamma^{\frac{1}{2}} \begin{pmatrix} \alpha_3 \\ \alpha_4 \end{pmatrix}, \quad (6.17)$$

with

$$\partial_z \nu = (\partial_z \nu_1, \partial_z \nu_2)^t = (\partial_z \mu)(z, h) + (\partial_z \tau)(z, \tilde{\alpha}(\tilde{\beta}, \alpha'), \alpha', h, \delta)$$

where  $\partial_z \tau$  satisfies the estimate given in (6.8). Furthermore, (6.12) and (6.13) imply that

$$L(d\tilde{\alpha}) = \delta^{-4} \left( \sqrt{\det A} + \mathcal{O} \left( \delta h^{-\frac{3}{2}} \right) \right)^{-2} L(d\tilde{\beta}) =: J(\tilde{\beta}, \alpha') L(d\tilde{\beta}) \quad (6.18)$$

By performing this change of variables in the integral  $I_1$  and by picking  $\varepsilon > 0$  small enough as above, we get that  $I_1$  is equal to

$$\lim_{\varepsilon \searrow 0} \pi^{-N} \iint_{\substack{\tilde{\beta} \in D(0, K\varepsilon) \times D(0, K\varepsilon) \\ (\tilde{\alpha}(\tilde{\beta}, \alpha'), \alpha') \in B(0, R) \\ \|\alpha'\|_{\mathbb{C}^{N-2}} \leq R_0}} H_\varepsilon^\delta(z_1, z_2, \tilde{\alpha}(\tilde{\beta}, \alpha'), \alpha'; h) e^{-\Phi(\tilde{\beta}, \alpha')} J(\tilde{\beta}, \alpha') L(d\alpha') L(d\tilde{\beta}),$$

where

$$\Phi(\tilde{\beta}, \alpha') := \tilde{\alpha}(\tilde{\beta}, \alpha')^* \cdot \tilde{\alpha}(\tilde{\beta}, \alpha') + (\alpha')^* \cdot \alpha'.$$

The integrand of  $I_1$  depends continuously on  $\tilde{\beta}$ . Hence, by performing the limit  $\varepsilon \rightarrow 0^+$ , we get

$$I_1(z, h) = \pi^{-N} \int_{\substack{(\tilde{\alpha}(0, \alpha'), \alpha') \in B(0, R) \\ \|\alpha'\|_{\mathbb{C}^{N-2}} \leq R_0}} H_0^\delta(z_1, z_2, \tilde{\alpha}(0, \alpha'), \alpha'; h) e^{-\Phi(0, \alpha')} J(0, \alpha') L(d\alpha') \quad (6.19)$$

with

$$H_0^\delta(z_1, z_2, \tilde{\alpha}(0, \alpha'), \alpha'; h) = |F_3(z, 0, \alpha') F_4(z, 0, \alpha')|^2.$$

Using (6.16), one computes that

$$\Phi(0, \alpha') = \frac{1}{\delta^2} \nu^* A^{-1} \nu + (\alpha')^* \cdot \alpha'$$

and, using (6.17), we get

$$\begin{pmatrix} F_3^\delta(z, \tilde{\alpha}(0, \alpha'), \alpha') \\ F_4^\delta(z, \tilde{\alpha}(0, \alpha'), \alpha') \end{pmatrix} = \partial_z \nu - B^* A^{-1} \nu - \delta \Gamma^{\frac{1}{2}} \begin{pmatrix} \alpha_3 \\ \alpha_4 \end{pmatrix}, \quad (6.20)$$



where  $\nu = \nu(z, \tilde{\alpha}(0, \alpha'), \alpha', h, \delta)$ . Using (6.8), (6.9) and (6.14) one computes that

$$\|\tilde{\alpha}(0, \alpha')\|^2 = \frac{1}{\delta^2} \nu^* A^{-1} \nu \leq C h^{-\frac{1}{5}} \left[ \mathcal{O}\left(\delta^{-2} e^{-\frac{2S}{h}}\right) + \mathcal{O}(\delta^2 h^{-5}) \right], \quad (6.21)$$

where the constant  $C > 0$  comes from the upper bound of  $\|A^{-1/2}\|$  given in (6.14). By the Hypothesis (1.11), we conclude that

$$\|\tilde{\alpha}(0, \alpha')\|^2 \ll h^{-\frac{1}{5}}.$$

which implies that  $(\tilde{\alpha}(0, 0, \alpha'), \alpha') \in B(0, R)$  for all  $\alpha'$  with  $\|\alpha'\|_{\mathbb{C}^{N-2}} \leq R_0$ . Hence,

$$I_1(z, h) = \pi^{-N} \int_{\|\alpha'\|_{\mathbb{C}^{N-2}} \leq R_0} |F_3(z, 0, \alpha') F_4(z, 0, \alpha')|^2 e^{-\Phi(0, \alpha')} J(0, \alpha') L(d\alpha'). \quad (6.22)$$

Next, we want to apply a multi-dimensional version of the mean value theorem for integrals to (6.22). Indeed, let  $U \subset \mathbb{R}^n$  be open, relatively compact and path-connected, it then holds true that for a continuous function  $f : \overline{U} \rightarrow \mathbb{R}$  and a positive integrable function  $g : \overline{U} \rightarrow \mathbb{R}$ , there exists a  $y \in \overline{U}$  such that

$$f(y) \int_U g(x) dx = \int_U f(x) g(x) dx.$$

Hence, the mean value theorem applied to (6.22) yields that

$$I_1(z, h) = \pi^{-N} J e^{-\frac{\tilde{\nu}^* A^{-1} \tilde{\nu}}{\delta^2}} \int_{\|\alpha'\|_{\mathbb{C}^{N-2}} \leq R_0} |F_3(z, 0, \alpha') F_4(z, 0, \alpha')|^2 e^{-\alpha' \overline{\alpha'}} L(d\alpha').$$

Here,  $J$  denotes the evaluation of the Jacobian  $J(0, \alpha')$  (cf. (6.18)) at the intermediate point for  $\alpha'$  given by mean value theorem. Note that  $J$  depends smoothly on  $z_1$  and  $z_2$  because  $\tau$  and  $A$  do.

Similarly,  $\tilde{\nu}$  above denotes the evaluation of the function  $\nu(z, \tilde{\alpha}(0, \alpha'), \alpha', h, \delta)$  at the intermediate point for  $\alpha'$  given by mean value theorem. It depends smoothly on  $z_1$  and  $z_2$  because  $\mu$  and  $\tau$  do. Moreover, using (6.8), we see that it satisfies

$$\tilde{\nu} = \begin{pmatrix} E_{-+}(z_1) \\ E_{-+}(z_2) \end{pmatrix} + \mathcal{O}\left(\delta^2 h^{-\frac{5}{2}}\right).$$

It remains to study the integral

$$\tilde{I}_1(z, h) := \pi^{-N} \int_{\|\alpha'\|_{\mathbb{C}^{N-2}} \leq R_0} |F_3(z, 0, \alpha') F_4(z, 0, \alpha')|^2 e^{-\alpha' \overline{\alpha'}} L(d\alpha'). \quad (6.23)$$

Define the linear forms

$$l_1(\alpha') = [\Gamma^{\frac{1}{2}}]_{11} \alpha_3 + [\Gamma^{\frac{1}{2}}]_{12} \alpha_4, \quad l_2(\alpha') = [\Gamma^{\frac{1}{2}}]_{21} \alpha_3 + [\Gamma^{\frac{1}{2}}]_{22} \alpha_4.$$

Using (6.20), we get that

$$\begin{aligned} F_3(z, 0, \alpha') &= (\partial_z \nu - B^* A^{-1} \nu)_1 - \delta l_1(\alpha') = \mathcal{O}\left(h^{-\frac{3}{5}} e^{-\frac{S}{h}} + \delta^2 h^{-\frac{37}{10}}\right) - \delta l_1(\alpha'), \\ F_4(z, 0, \alpha') &= (\partial_z \nu - B^* A^{-1} \nu)_2 - \delta l_2(\alpha') = \mathcal{O}\left(h^{-\frac{3}{5}} e^{-\frac{S}{h}} + \delta^2 h^{-\frac{37}{10}}\right) - \delta l_2(\alpha'), \end{aligned} \quad (6.24)$$

where the error estimate is uniform in  $\alpha'$ , for  $\|\alpha'\|_{\mathbb{C}^{N-2}} \leq R_0$ . In the last equation we used (6.8), (6.9), (6.14) and the fact that the Hilbert-Schmidt norm of  $B^*$  is  $\leq \frac{1}{h\mathcal{O}(1)}$  which follows from the fact that elements of the matrix  $B^*$  are bounded by a term of order  $h^{-1}$ .

By Proposition 36, one gets that the Hilbert-Schmidt norm of  $\Gamma^{\frac{1}{2}}$  is bounded, indeed one has that

$$\|\Gamma^{\frac{1}{2}}\|_{\text{HS}} = \sqrt{\text{tr } \Gamma} \leq \mathcal{O}(h^{-\frac{1}{2}}).$$

Hence, the linear forms  $l_i(\alpha')$ ,  $i = 1, 2$ , satisfy

$$|l_i(\alpha')| \leq \mathcal{O}(h^{-\frac{1}{2}}) \|(\alpha_3, \alpha_4)\|.$$

Using (6.24), we compute that

$$|F_3(z, 0, \alpha') F_4(z, 0, \alpha')|^2 = \delta^4 \left( |l_1(\alpha') l_2(\alpha')|^2 + \mathcal{O}\left(e^{-\frac{1}{C\hbar}} + \delta h^{-\frac{52}{10}}\right) \sum_{j=0}^3 \|(\alpha_3, \alpha_4)\|^j \right). \quad (6.25)$$

Here we used as well that by Hypothesis 2, we have that  $\mathcal{O}(\delta^{-1} e^{-\frac{S}{\hbar}}) = \mathcal{O}(e^{-\frac{1}{C\hbar}})$ . Observe that since  $C > C_0 > C_1 > 0$ , see the discussion before (6.10), we have that for  $k = 0, \dots, 4$

$$\pi^{-N} \int_{\|\alpha'\|_{\mathbb{C}^{N-2}} \geq R_0} \|(\alpha_3, \alpha_4)\|^k e^{-\alpha' \overline{\alpha'}} L(d\alpha') \leq \mathcal{O}\left(e^{-\frac{D}{\hbar^2}}\right). \quad (6.26)$$

Technically this holds true if the difference  $C_0 - C_1 > 0$  is assumed to be sufficiently large. Notice that we have room for that if we take  $C > 0$  in (1.8) large enough to begin with and choose  $C_0$  in the discussion before (6.10) sufficiently large.

Extend the function  $|F_3(z, 0, \alpha') F_4(z, 0, \alpha')|^2$  in the variables  $\alpha'$  to the whole of  $\mathbb{C}^{N-2}$  by a function such that (6.25) holds for all  $\alpha' \in \mathbb{C}^{N-2}$ . Hence, by (6.26), (6.23)

$$\tilde{I}_1(z, h) = \delta^4 \pi^{-N} \int_{\mathbb{C}^{N-2}} |l_1(\alpha') l_2(\alpha')|^2 e^{-\alpha' \overline{\alpha'}} L(d\alpha') + \mathcal{O}\left(\delta^4 e^{-\frac{1}{C\hbar}} + \delta^5 h^{-\frac{52}{10}}\right).$$

Integration by parts yields that

$$\pi^{-N} \int_{\mathbb{C}^{N-2}} |l_1(\alpha') l_2(\alpha')|^2 e^{-\alpha' \overline{\alpha'}} L(d\alpha') = \pi^{-4} \int_{\mathbb{C}^2} e^{-\tilde{\alpha} \overline{\tilde{\alpha}}} \prod_{k=1}^2 l_k(\overline{\partial_{\tilde{\alpha}}}) \left( \prod_{n=1}^2 \overline{l_n(\tilde{\alpha})} \right) L(d\tilde{\alpha}).$$

Note that for any permutation  $\sigma \in S_n$ , where  $S_n$  is the symmetric group, we have that  $(l_i | l_{\sigma(i)}) = \Gamma_{i\sigma(i)}$ . Thus, in view of (3.15), we have that

$$\prod_{k=1}^2 l_k(\overline{\partial_{\tilde{\alpha}}}) \left( \prod_{n=1}^2 \overline{l_n(\tilde{\alpha})} \right) = \sum_{\sigma \in S_2} (l_1 | l_{\sigma(1)}) (l_2 | l_{\sigma(2)}) = \text{perm } \Gamma.$$

We conclude that

$$I_1(z, h) = \frac{\text{perm } \Gamma + \mathcal{O}\left(e^{-\frac{1}{C\hbar}} + \delta h^{-\frac{52}{10}}\right)}{\pi^2 \left( \sqrt{\det A} + \mathcal{O}\left(\delta h^{-\frac{3}{2}}\right) \right)^2},$$

where we used the fact that  $\det A \geq \frac{h^{\frac{1}{5}}}{\mathcal{O}(1)}$  for  $1/C \geq |z - w| \gg h^{3/5}$ , see Proposition 33, to obtain the last equality.

**The integral  $I_2$**  In this step we will estimate the second integral of equation (6.11). Therefore, we will increase the space of integration

$$\begin{aligned} & \pi^{-N} \int_{\substack{B(0,R) \\ R_0 < \|\alpha'\|_{\mathbb{C}^{N-2}} < R}} \prod_{k=1}^2 \varepsilon^{-2} \chi\left(\frac{F_k(z, \alpha)}{\varepsilon}\right) |\partial_{z_k} F_k(z, \alpha)|^2 e^{-\alpha \bar{\alpha}} L(d\alpha) \\ & \leq \pi^{-N} \int_{\substack{B(0,2R) \\ R_0 < \|\alpha'\|_{\mathbb{C}^{N-2}} < 2R_0}} \prod_{k=1}^2 \varepsilon^{-2} \chi\left(\frac{F_k(z, \alpha)}{\varepsilon}\right) |\partial_{z_k} F_k(z, \alpha)|^2 e^{-\alpha \bar{\alpha}} L(d\alpha) =: W_\varepsilon. \end{aligned}$$

It is easy to see that Lemma 38 holds true for the set  $B(0, 2R) \cap \{R_0 < \|\alpha'\|_{\mathbb{C}^{N-2}} < 2R_0\}$ . Therefore, we can proceed as for the integral  $I_1$ : perform the same change of variables and perform the limit of  $\varepsilon \rightarrow 0$ . As for  $I_1$ , the integrand remains bounded by at most a finite power of  $h^{-1}$  which then yields that

$$\lim_{\varepsilon \rightarrow 0} W_\varepsilon = \mathcal{O}\left(e^{-\frac{D}{h^2}}\right),$$

where the exponential decay comes from the fact that  $R_0 < \|\alpha'\|_{\mathbb{C}^{N-2}}$ . Therefore,

$$\int_{\mathbb{C}^2} \varphi_1(z_1) \varphi_2(z_2) d\nu(z_1, z_2) = \int_{\mathbb{C}^2} \varphi_1(z_1) \varphi_2(z_2) D(z, h) L(dz_1 dz_2)$$

with

$$D(z, h, \delta) = \frac{\text{perm } \Gamma + \mathcal{O}\left(e^{-\frac{1}{Ch}} + \delta h^{-\frac{52}{10}}\right)}{\pi^2 \left(\sqrt{\det A} + \mathcal{O}\left(\delta h^{-\frac{3}{2}}\right)\right)^2} + \mathcal{O}\left(e^{-\frac{D}{h^2}}\right). \quad \square$$

## 7. PROOF OF THE MAIN RESULTS

Using the above results, in particular Propositions 17 and 37, we can now prove Theorem 5, Theorem 7, Theorem 8 and Corollary 10.

*of Theorem 5.* The result follows directly from Proposition 17 with the density  $D$  given by Proposition 37 and by Proposition 30.  $\square$

*of Theorem 7.* First, let us treat the case of the long range interaction: we suppose that  $|z - w| \gg (h \ln h^{-1})^{\frac{1}{2}}$ . Here, we have that for any power  $N > 1$  the term

$$\left(\frac{\sigma_h(z, w)|z - w|^2}{4h}\right)^N e^{-K(z, w)}$$

remains bounded. Using that  $\sinh K(z, w) \geq \mathcal{O}(h^{-C}) > 0$  with  $C \gg 1$  and using that  $\sigma_h(z, z) = \sigma(z) + \mathcal{O}(h)$ , it follows that

$$D^\delta(z, w; h) = \frac{\sigma(z)\sigma(w) + \mathcal{O}(h)}{(2h\pi)^2} \left(1 + \mathcal{O}\left(\delta h^{-\frac{8}{5}}\right)\right).$$

Next, we consider the case where  $h^{\frac{4}{7}} \ll |z - w| \ll h^{\frac{1}{2}}$ . Recall from Theorem 5 that

$$D^\delta(z, w; h) = \frac{\Lambda(z, w)}{(2\pi h)^2 (1 - e^{-2K(z, w)})} \left(1 + \mathcal{O}\left(\delta h^{-\frac{8}{5}}\right)\right) + \mathcal{O}\left(e^{-\frac{D}{h^2}}\right) \quad (7.1)$$

with  $\Lambda(z, w; h)$  equal to

$$\begin{aligned} & \sigma_h(z, z)\sigma_h(w, w) + \sigma_h(z, w)^2(1 + \mathcal{O}(|z - w|))e^{-2K(z, w)} + \mathcal{O}\left(h^\infty + \delta h^{-\frac{32}{10}}\right) \\ & + \frac{\sigma_h(z, w)^2(1 + \mathcal{O}(|z - w|))}{e^{K(z, w)} \sinh K(z, w)} \left( \left( \frac{\sigma_h(z, w)|z - w|^2}{4h} \right)^2 2 \coth K(z, w) - \frac{\sigma_h(z, w)|z - w|^2}{h} \right). \end{aligned}$$

Similarly to (5.7), we have that  $\sigma_h(z, z) = \sigma_h(z, w)(1 + \mathcal{O}(|z - w|))$ . We start by considering the first term in (7.1):

$$\frac{\Lambda(z, w)}{(2\pi h)^2 (1 - e^{-2K(z, w)})}. \quad (7.2)$$

Set  $\sigma_h = \sigma_h(z, w)$ . Using the Taylor expansions of the functions  $\sinh x$ ,  $\coth x$  and  $e^{-x}$ , one computes, that (7.2) is equal to

$$\begin{aligned} & \frac{1}{h\pi^2 \sigma_h |z - w|^2 \left(1 + \mathcal{O}\left(\frac{|z - w|^2}{h}\right)\right)} \left[ \sigma_h^2 (1 + \mathcal{O}(|z - w|)) - \frac{\sigma_h^3 |z - w|^2}{4h} (1 + \mathcal{O}(|z - w|)) \right. \\ & + \frac{\sigma_h^4 |z - w|^4}{4^2 h^2} \left(1 + \mathcal{O}\left(\frac{|z - w|^2}{h}\right)\right) + \left\{ \frac{\sigma_h^4 |z - w|^4}{3 \cdot 4^4 h^2} \left(1 + \mathcal{O}\left(\frac{|z - w|^4}{h^2}\right)\right) - 1 \right\} \\ & \cdot \frac{\sigma_h^2 \left(1 - \frac{\sigma_h |z - w|^2}{4h} (1 + \mathcal{O}(|z - w|)) + \frac{\sigma_h^2 |z - w|^4}{2 \cdot 4^2 h} \left(1 + \mathcal{O}\left(\frac{|z - w|^2}{h}\right)\right)\right)}{1 + \mathcal{O}(|z - w|) + \frac{\sigma_h^2 |z - w|^4}{4^2 \cdot 6h} \left(1 + \mathcal{O}\left(\frac{|z - w|^2}{h}\right)\right)} + \mathcal{O}\left(h^\infty + \delta h^{-\frac{32}{10}}\right) \Big] \end{aligned}$$

which simplifies to

$$\Lambda(z, w; h) = \frac{\sigma_h^3 |z - w|^2}{(4\pi)^2 h^3} \left(1 + \mathcal{O}\left(\frac{|z - w|^2}{h}\right)\right).$$

Hence,

$$D^\delta(z, w; h) = \frac{\sigma_h^3 |z - w|^2}{(4\pi)^2 h^3} \left(1 + \mathcal{O}\left(\frac{|z - w|^2}{h} + \delta h^{-\frac{8}{5}}\right)\right)$$

which concludes the proof.  $\square$

of Theorem 8. Using that  $\sigma_h(z, w_0) = \sigma_h(z, z)(1 + \mathcal{O}(|z - w_0|))$  (cf. (5.7) and (5.8)), the result of Theorem 8 follows from Theorems 5 and 7.  $\square$

of Corollary 10. Let  $W \subseteq \{(z, w) \in \mathbb{C}^2; z \neq w\}$  be compact. Recall from the discussion at the beginning of Section 2.3 that, for  $h > 0$  small enough,

$$\tilde{\kappa}_h(z, w) := \kappa^\delta(z_0 + d_0^{-1/2} z, z_0 + d_0^{-1/2} w; h),$$

is well defined for all  $(z, w) \in W$ , where  $d_0 := d(z_0; h) \asymp h^{-1}$ , see (2.6), (2.7). Using Theorems 5 and 8 we see that

$$\begin{aligned} & \kappa^\delta(z_0 + d_0^{-1/2}z, z_0 + d_0^{-1/2}w; h) \\ &= \frac{(1 + \mathcal{O}(h))((\sinh^2 K + (1 + \mathcal{O}(h^{1/2}))K^2) \cosh K - (1 + \mathcal{O}(h^{1/2}))2K \sinh K)}{\sinh^3 K} \\ & \quad + \frac{\mathcal{O}\left(h^\infty + \delta h^{-\frac{32}{10}}\right)}{(1 - e^{-2K})} + \mathcal{O}\left(e^{-\frac{D}{h^2}}\right), \end{aligned}$$

with

$$\begin{aligned} K &= K(z_0 + d_0^{-1/2}z, z_0 + d_0^{-1/2}w; h) \\ &= \sigma_h(z_0 + d_0^{-1/2}z, z_0 + d_0^{-1/2}w) \frac{|z - w|^2}{4hd_0} (1 + \mathcal{O}(h^{1/2})) \\ &= \frac{\pi}{2} |z - w|^2 (1 + \mathcal{O}(h^{1/2})), \end{aligned}$$

where the error estimates are uniform in  $W$ . Here, we used as well that  $d_0 = (2\pi h)^{-1} \sigma(z_0) (1 + \mathcal{O}(h))$ , cf. (2.6), and that by Taylor expansion  $\sigma_h(z_0 + d_0^{-1/2}z, z_0 + d_0^{-1/2}w) = \sigma(z_0) (1 + \mathcal{O}(h^{1/2}))$ . Taking the limit  $h \rightarrow 0^+$  we conclude the statement of the Corollary.  $\square$

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